Random Modular Symbols

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Explicit Methods in Number Theory
Oberwolfach

Goal

$\mathrm{St}(\mathbb{Q}^2;\mathbb{Z})$ is isomorphic to modular symbols.
$H_0(\Gamma,\mathrm{St}(\mathbb{Q}^2;\mathbb{C}))\simeq H^1(\Gamma;\mathbb{C})$ computes weight 2 modular forms for Γ.

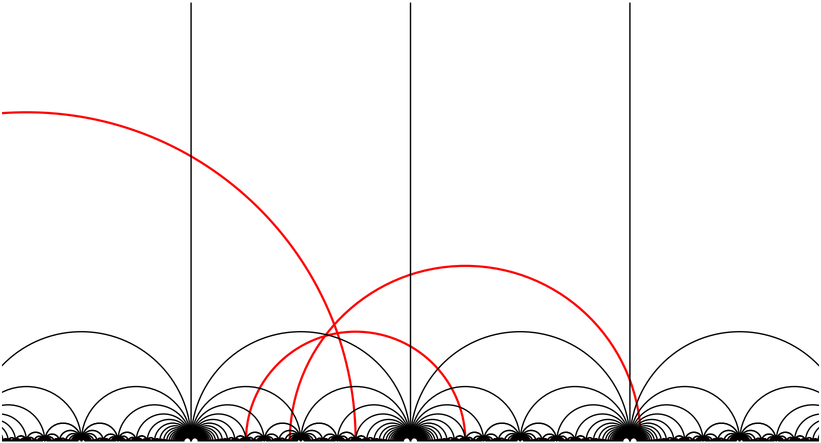
Understand the image of the set of modular symbols in the homology.

- Is it finite or infinite?
- Does it have any structure?
- How are the symbols distributed among Hecke eigenspaces?

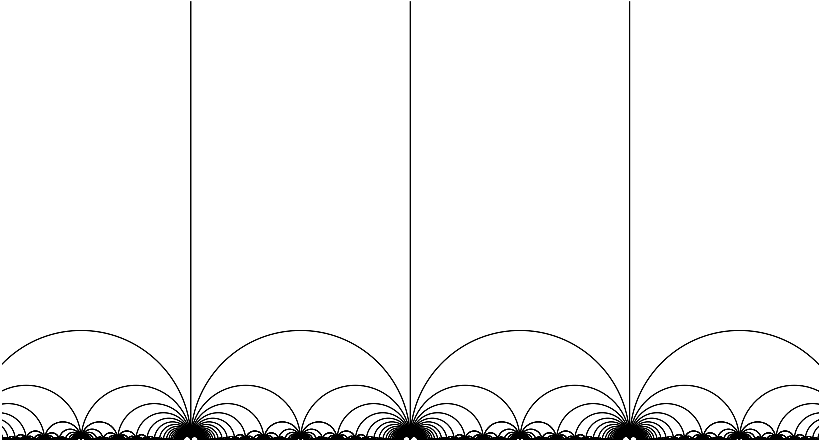
Abelian group generated by $[v, w]$ where v, w are elements in the projective space $\mathbb{P}^1(\mathbb{Q})$ such that

1. $[v, w] = -[w, v]$ for all $v, w \in \mathbb{P}^1(\mathbb{Q})$;
2. $[v, w] = [v, x] + [x, w]$ for all $v, w, x \in \mathbb{P}^1(\mathbb{Q})$;
3. $[a, b] = 0$ for all $a, b \in \mathbb{Q}^2$ such that the determinant of $[a, b]$ equals 0.

Modular symbols

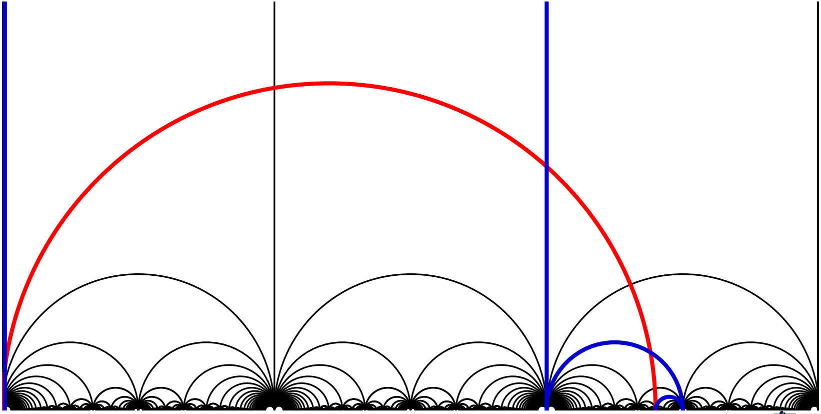


Unimodular symbols: A small generating set



Reduction algorithm: Manin's trick

There is a reduction algorithm using continued fractions to express any modular symbol as a $\mathbb{Z}$-linear combination of unimodular symbols.



Notation

Fix prime level N.

$$\Gamma = \Gamma_0^{\pm}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}$$

$$V = H^1(\Gamma, \mathbb{Q})$$

Types

Decompose V into $\mathbb{Q}\mathcal{H}$-irreducible subspaces:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus E, \quad \text{with} \quad \pi_i \colon V \to V_i.$$

Definition

For $v \in V$, define the *type of v* to be

$$t(v) = (t_1, t_2, \ldots, t_k, t_E),$$

where

$$t_\alpha = \begin{cases} 1 & \text{if } \pi_\alpha(v) \neq 0, \\ 0 & \text{if } \pi_\alpha(v) = 0. \end{cases}$$

We examined in detail the types for prime $N < 100$ for 6,496,360,000 modular symbols.

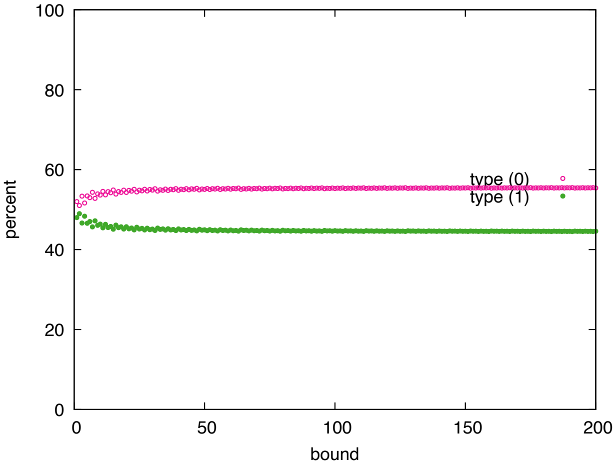


Figure: $N = 2$

One eigenspace $V = E$: $N \in \{2, 3, 5, 7, 13\}$

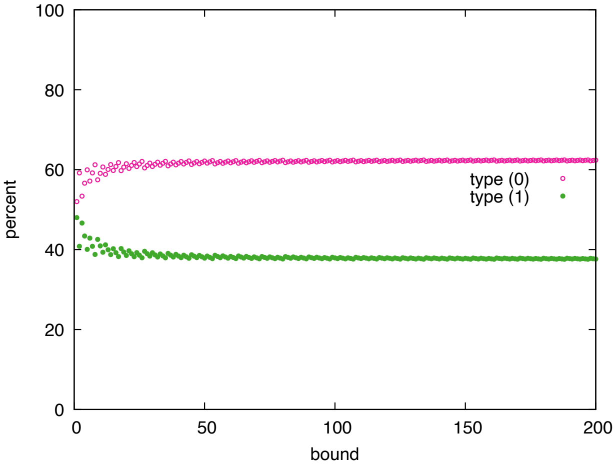


Figure: $N = 3$

One eigenspace $V = E$: $N \in \{2, 3, 5, 7, 13\}$

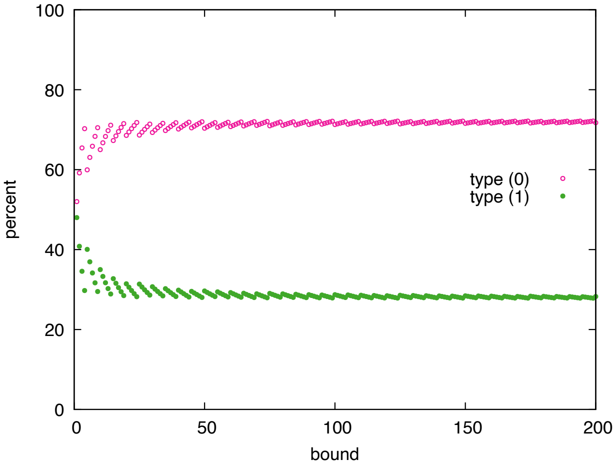


Figure: $N = 5$

One eigenspace $V = E$: $N \in \{2, 3, 5, 7, 13\}$

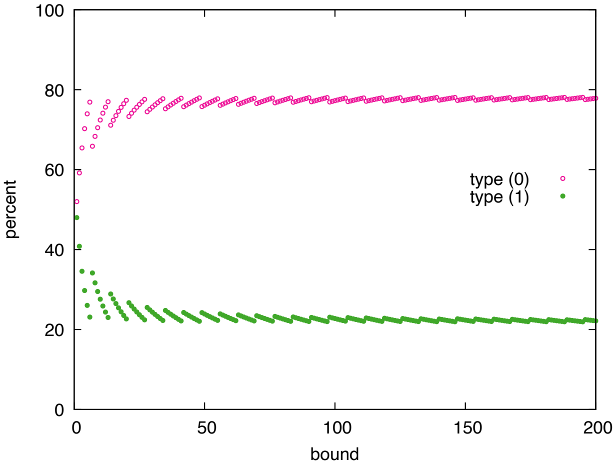


Figure: $N = 7$

One eigenspace $V = E$: $N \in \{2, 3, 5, 7, 13\}$

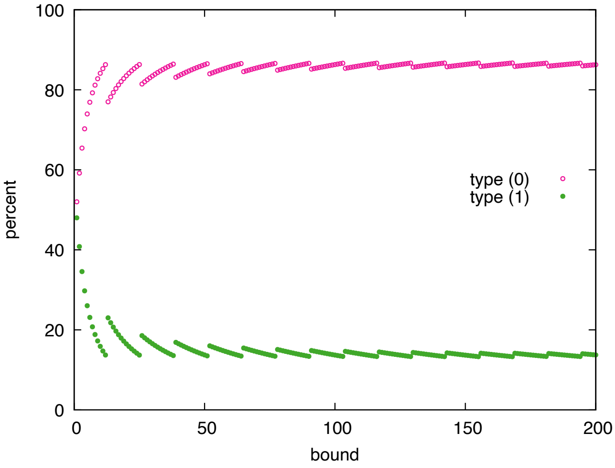


Figure: $N = 13$

What does a "random" modular symbol look like?

Theorem

Let $a, b, p, q \in \mathbb{Z}$ *with* $\begin{bmatrix} a \\ b \end{bmatrix}$ *and* $\begin{bmatrix} p \\ q \end{bmatrix}$ *not equal to* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *be chosen from a "rectangular box", and suppose it has type*

$$t([a/b, p/q]_\Gamma) = (t_1, t_2, \ldots, t_k, t_E).$$

Then as the box grows to infinity, the probability that $t_E = 0$ *is* $\frac{1+N^2}{(1+N)^2}$, *and the probability that* $t_E = 1$ *is* $\frac{2N}{(1+N)^2}$.

$N = 37$

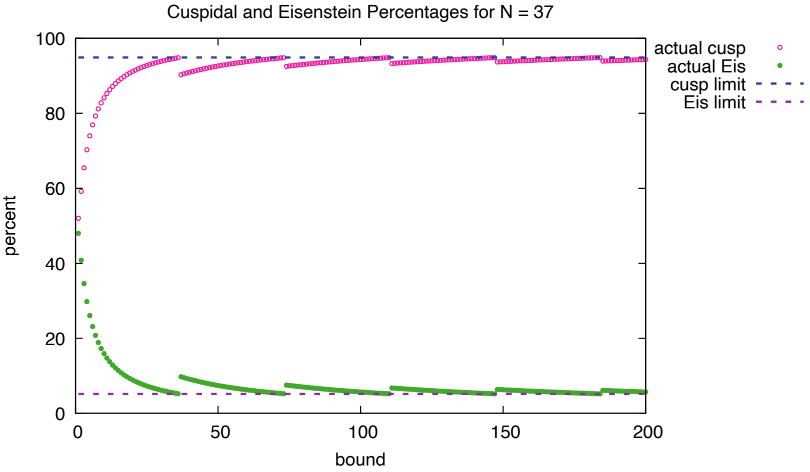


Figure: The percentage of cuspidal and noncuspidal (Eis) modular symbols observed for level 37 as a function of box size.

Set of modular symbols and cuspidal symbols

$$U = \{[x, y]_\Gamma \mid x, y \in \mathbb{P}^1(\mathbb{Q})\} \subseteq V$$
$$U' = \{[x, \gamma x]_\Gamma \mid x \in \mathbb{P}^1(\mathbb{Q}), \gamma \in \Gamma\} \subseteq U$$

- U and U' are a priori just sets.
- U' elements are cuspidal classes.
- U generates V, and U' generates $H^1_{\text{cusp}}(\Gamma, \mathbb{Q})$.

Lattice of modular symbols and cuspidal symbols

Let $\Lambda \subset V$ be the $\mathbb{Z}$-lattice generated by U. Define Λ' analogously.

Theorem

1. *The image of the cuspidal modular symbols in V is the cuspidal modular symbol lattice,*

$$U' = \Lambda'.$$

2. *The image of the modular symbols in V is the union of three cosets in Λ/Λ',*

$$U = \Lambda' \cup ([e, f]_\Gamma + \Lambda') \cup (-[e, f]_\Gamma + \Lambda'),$$

where $e = 1/0$ and $f = 0/1$ in $\mathbb{P}^1(\mathbb{Q})$.

Proof of 1

Proof.

Note: $U' = \{[f, \gamma f]_\Gamma \mid \gamma \in \Gamma\}$. Show that U' is closed under negation and addition.

Negation:

$$-[f, \gamma f]_\Gamma = [\gamma f, f]_\Gamma = [f, \gamma^{-1} f]_\Gamma$$

Addition:

$$\begin{aligned}[f, \gamma f]_\Gamma + [f, \tau f]_\Gamma &= [\gamma^{-1} f, f]_\Gamma + [f, \tau f]_\Gamma \\ &= [\gamma^{-1} f, \tau f]_\Gamma \\ &= [f, \gamma\tau f]_\Gamma\end{aligned}$$

□

Proof of 2

Proof.

If $[x, y]_\Gamma \in U$ is not cuspidal, then

1. x is Γ-equivalent to f and y is Γ-equivalent to e; or
2. x is Γ-equivalent to e and y is Γ-equivalent to f.

Suppose $x = \gamma f$ and $y = \tau e$ for some $\gamma, \tau \in \Gamma$. Then

$$[x, y]_\Gamma + [e, f]_\Gamma = [\gamma f, \tau e]_\Gamma + [e, f]_\Gamma = [\tau^{-1}\gamma f, e]_\Gamma + [e, f]_\Gamma$$
$$= [\tau^{-1}\gamma f, f]_\Gamma = [f, \gamma^{-1}\tau f]_\Gamma.$$

It follows that

$$[x, y]_\Gamma = -[e, f]_\Gamma + [f, \sigma f]_\Gamma,$$

for some $\sigma \in \Gamma$, so $[x, y]_\Gamma \in -[e, f]_\Gamma + \Lambda'$.
A similar argument shows the other case. □

No obstruction for cuspidal types

Since $U' = \Lambda'$, every nontrivial cuspidal type occurs infinitely often. There is **no obstruction for purely cuspidal types.**

Eisenstein obstruction for non-cuspidal types

For $i = 1, 2, \ldots, k$, let $\Lambda_i \subset V_i$ be the lattice

$$\Lambda_i = \mathrm{span}_{\mathbb{Z}}\{\pi_i([x,y]_\Gamma) \mid [x,y]_\Gamma \in U\}.$$

Define Λ_i' similarly.

Theorem

Suppose $[\Lambda_i : \Lambda_i'] \neq 1$*, and let* $[x,y]_\Gamma \in V$ *have type*

$$t([x,y]_\Gamma) = (t_1, t_2, \ldots, t_k, t_E).$$

Then $t_E = 1$ *implies that* $t_i = 1$*.*

Eisenstein obstruction for non-cuspidal types

Key facts:

- For non-cuspidal $[x, y]_\Gamma$, $[x, y]_\Gamma = -[e, f]_\Gamma + [f, \sigma f]_\Gamma$.
- $[\Lambda_i \colon \Lambda_i'] = 1$ if and only if $\pi_i([e, f]_\Gamma) \in \Lambda_i'$.

Proof.

Suppose $t_E = 1$. Then $[x, y]_\Gamma$ is not cuspidal, and

$$\pi_i([x, y]_\Gamma) = \pi_i([f, \gamma f]_\Gamma) - \pi_i([e, f]_\Gamma), \quad \text{for some } \gamma \in \Gamma.$$

Since $[f, \gamma f]_\Gamma$ is cuspidal, we have $\pi_i([f, \gamma f]_\Gamma) \in \Lambda_i'$. If $[\Lambda_i \colon \Lambda_i'] \neq 1$, then $\pi_i([e, f]) \notin \Lambda_i'$, and so

$$\pi_i([x, y]_\Gamma) = \pi_i([f, \gamma f]_\Gamma) - \pi_i([e, f]_\Gamma) \neq 0.$$

□

Corollary

If $p \neq 2$ divides the index $[\Lambda_i : \Lambda_i']$, then there is a newform of level N and weight 2 whose Hecke eigenvalue a_ℓ is congruent modulo p to $\ell + 1$ for all ℓ not dividing N. Such a prime p divides $N - 1$.

$$N = 11, p = 5, a_\ell; \quad N = 23, p = 11, b_\ell, \beta = \tfrac{1}{2}(1 + \sqrt{5})$$

ℓ	2	3	5	7	11	13	...
a_ℓ	-2	-1	1	-2	1	4	...
a_ℓ mod 5	3	4	1	3	1	4	...
b_ℓ	$-\beta$	$2\beta - 1$	-2β	$-2\beta + 2$	$2\beta - 4$	3	...
b_ℓ mod $\mathfrak{p}_{11}$	3	4	6	8	1	3	...

Observations

For these prime levels $N < 10{,}000$, we observe from our data that

- the product of the indices divides $N-1$, i.e.,

$$\prod_{i=1}^{k}[\Lambda_i : \Lambda_i'] \mid (N-1);$$

- the quotient

$$Q = \frac{(N-1)}{\prod_i [\Lambda_i : \Lambda_i']}$$

  is a positive power of 2 times a nonnegative power of 3;

Two eigenspaces $V = V_1 \oplus E$: $N \in \{11, 17, 19, 23, 29, 31, 41, 47, 59\}$

Table: Lattice indices and the quotient $Q = (N-1)/[\Lambda_1 : \Lambda_1']$. Obstruction prevents type $(0, 1)$.

N	$\dim(V_1)$	$[\Lambda_1 : \Lambda_1']$	Q
11	1	5	2
17	1	2	2^3
19	1	3	$2 \cdot 3$
23	2	11	2
29	2	7	2^2
31	2	5	$2 \cdot 3$
41	3	5	2^3
47	4	23	2
59	5	29	2

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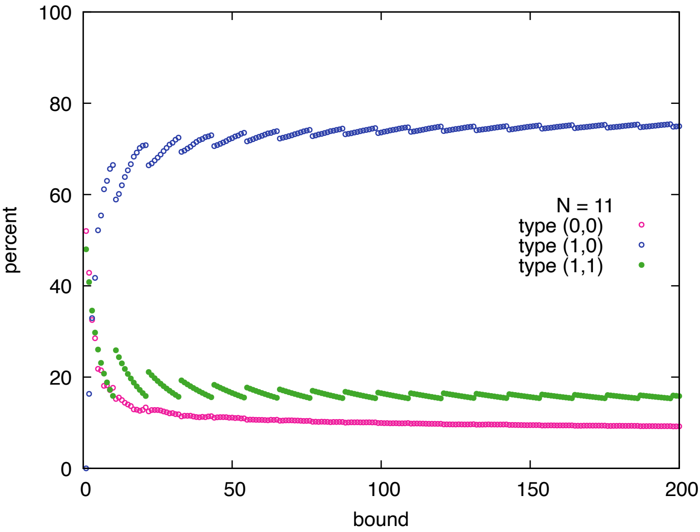
Two eigenspaces $V = V_1 \oplus E$: $N \in \{11, 17, 19, 23, 29, 31, 41, 47, 59\}$



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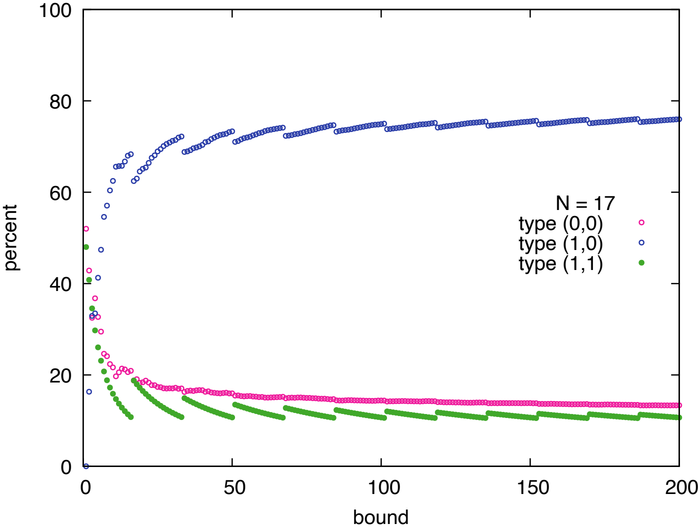

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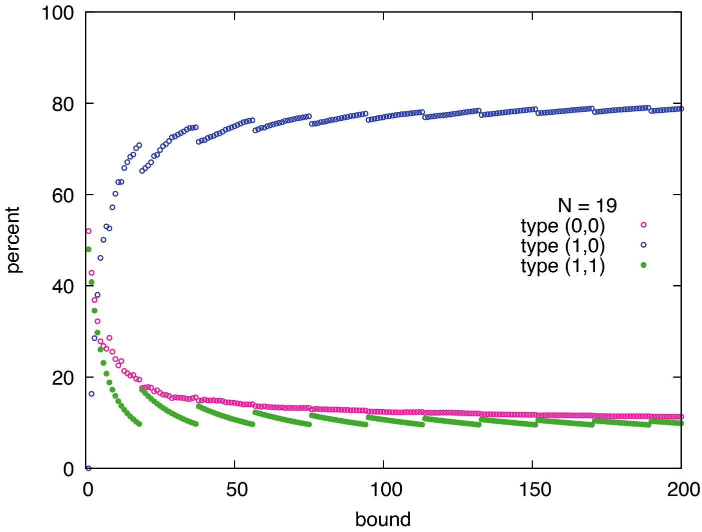

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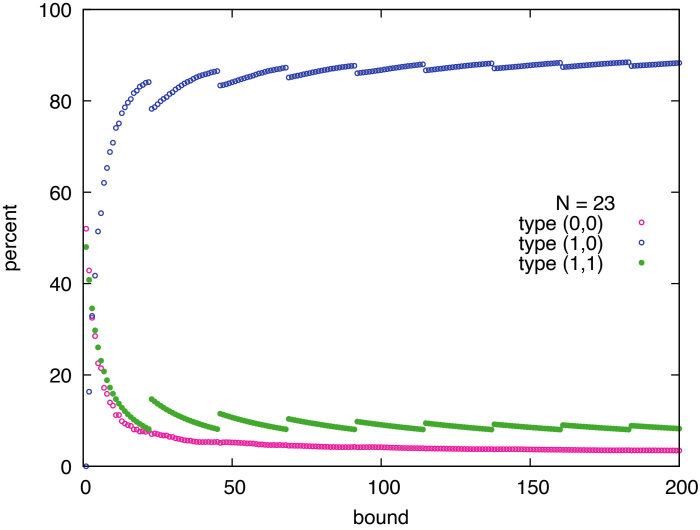

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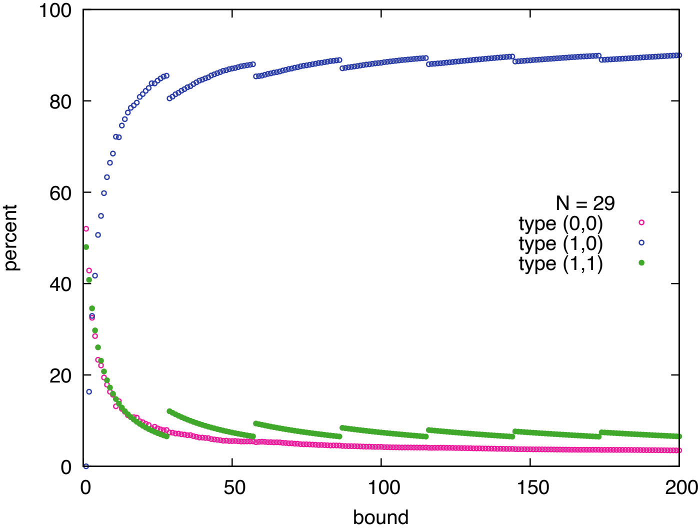

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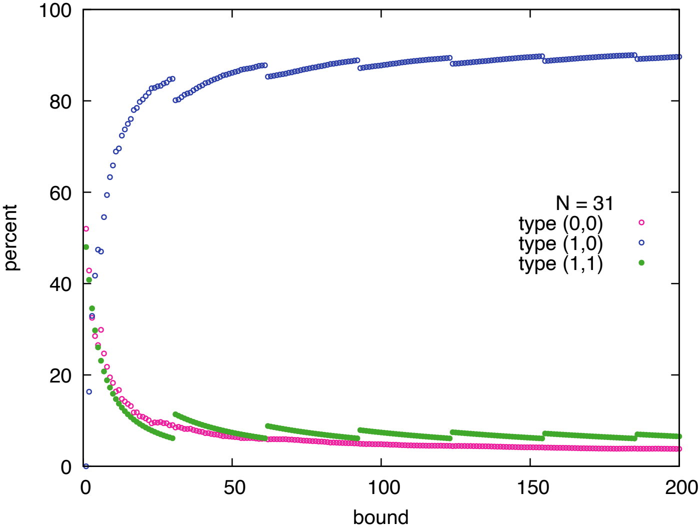

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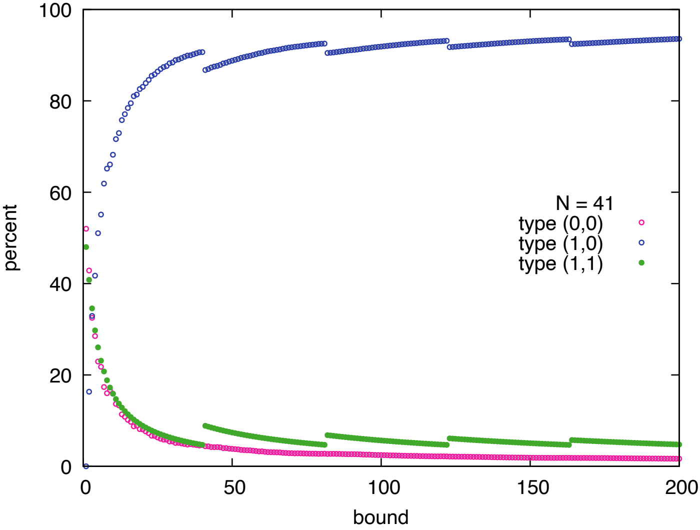

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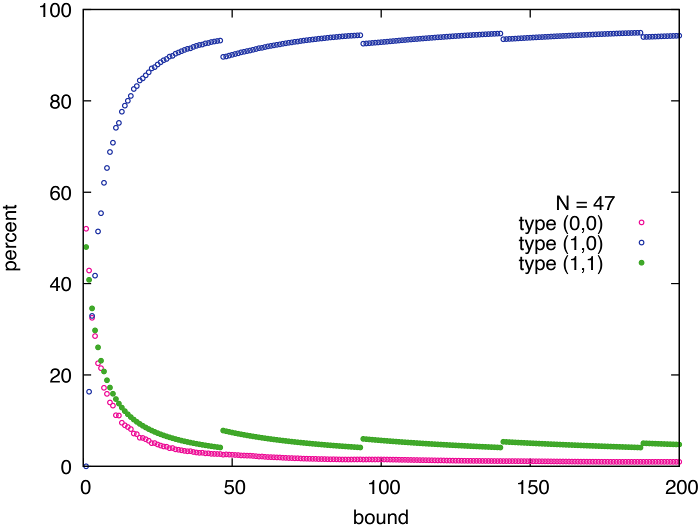

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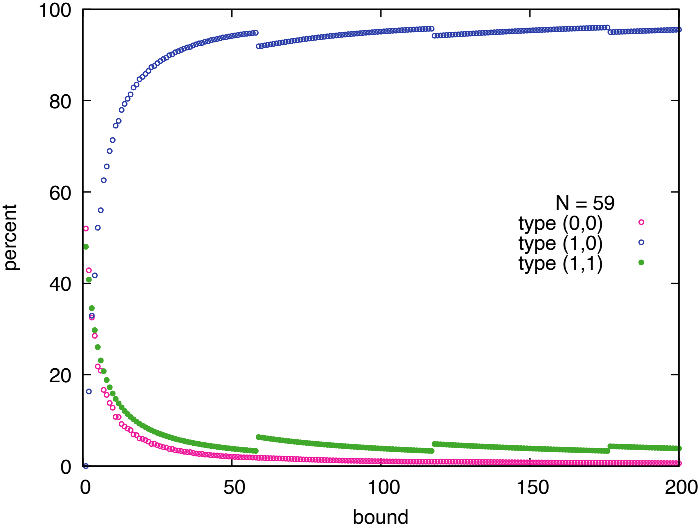

Three eigenspaces $V = V_1 \oplus V_2 \oplus E$: $N \in \{37, 43, 53, 61, 71, 79, 83, 97\}$

Table: Lattice indices, and the quotient $Q = (N-1)/\Pi_{i=1}^{2}[\Lambda_i : \Lambda_i']$. Two obstructions for $N = 71$.

N	$\dim(V_1)$	$\dim(V_2)$	$[\Lambda_1 : \Lambda_1']$	$[\Lambda_2 : \Lambda_2']$	Q
37	1	1	1	3	$2^2 \cdot 3$
43	1	2	1	7	$2 \cdot 3$
53	1	3	1	13	2^2
61	1	3	1	5	$2^2 \cdot 3$
71	3	3	7	5	2
79	1	5	1	13	$2 \cdot 3$
83	1	6	1	41	2
97	3	4	1	4	$2^3 \cdot 3$

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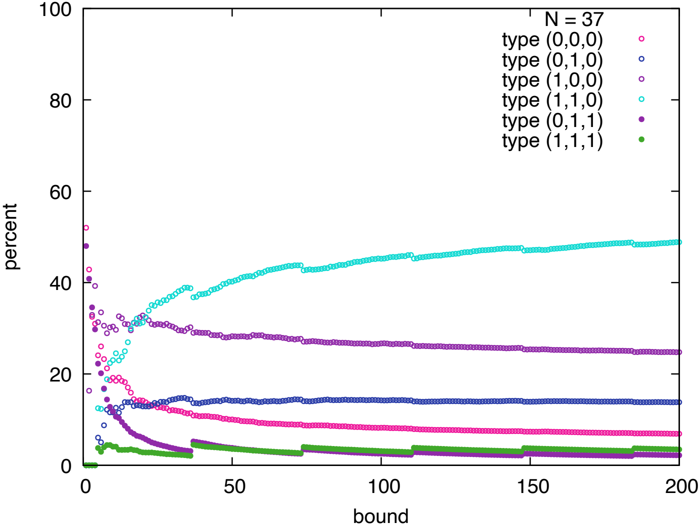

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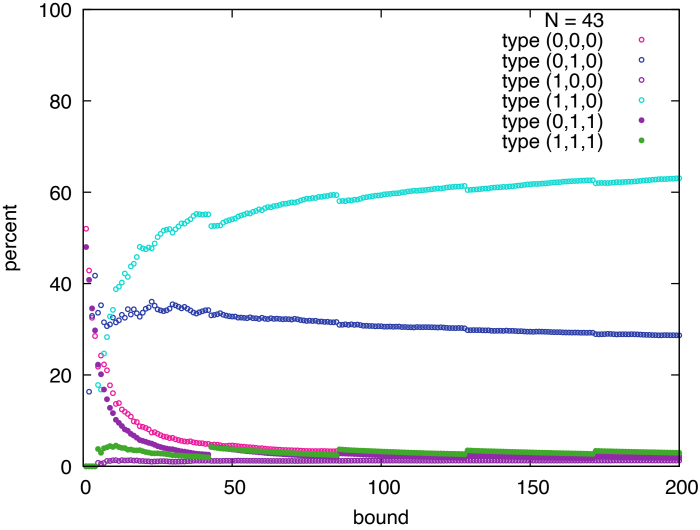

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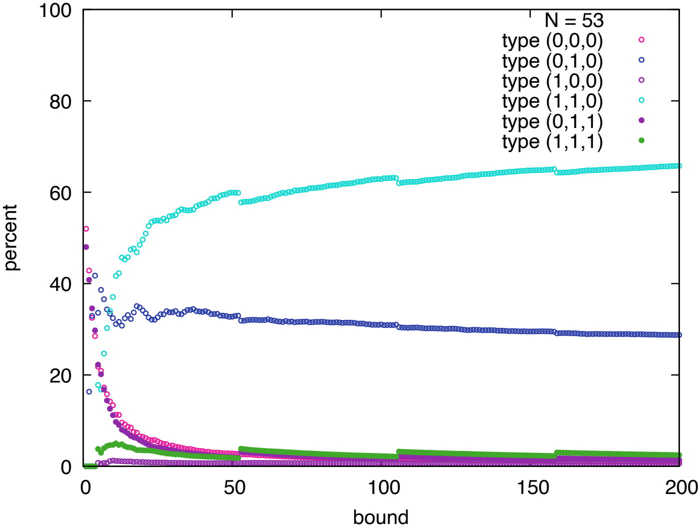

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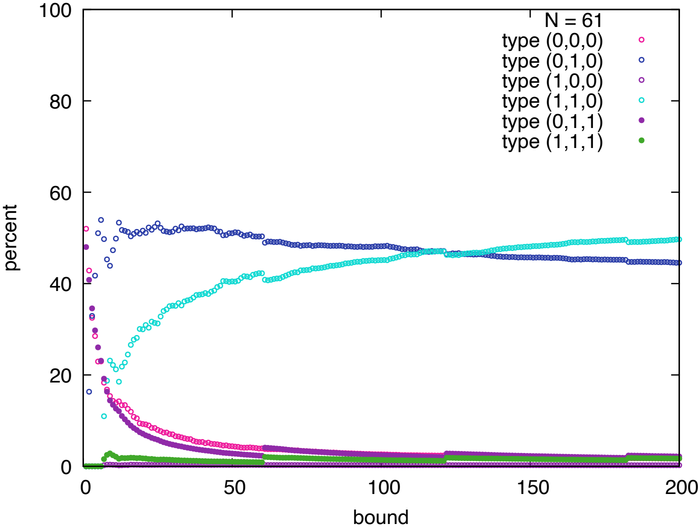

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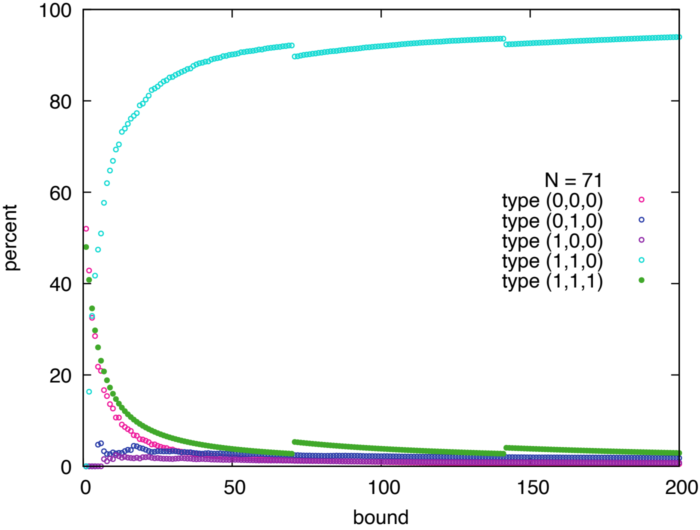

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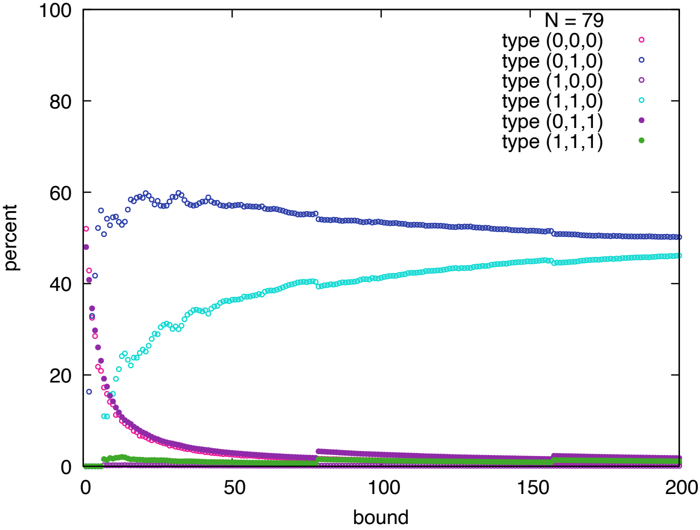

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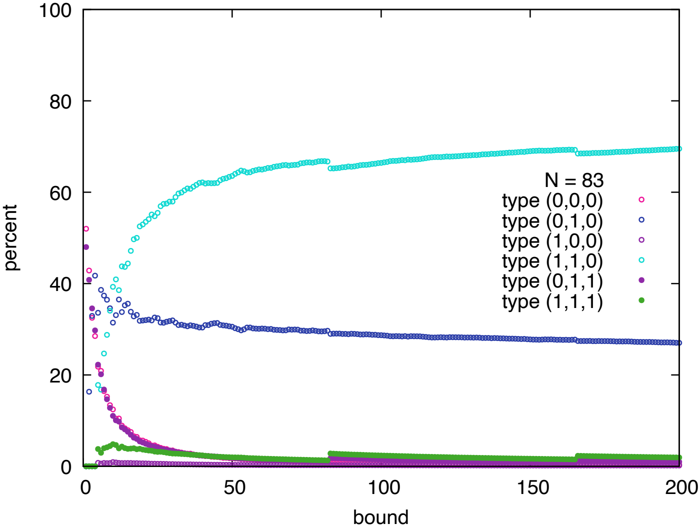

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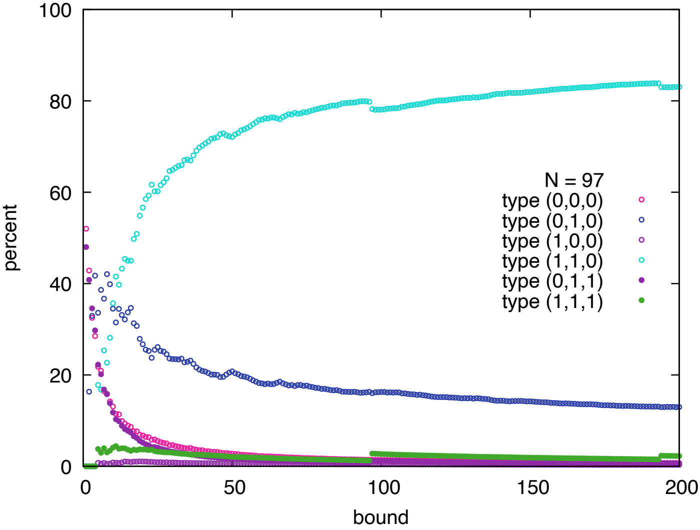
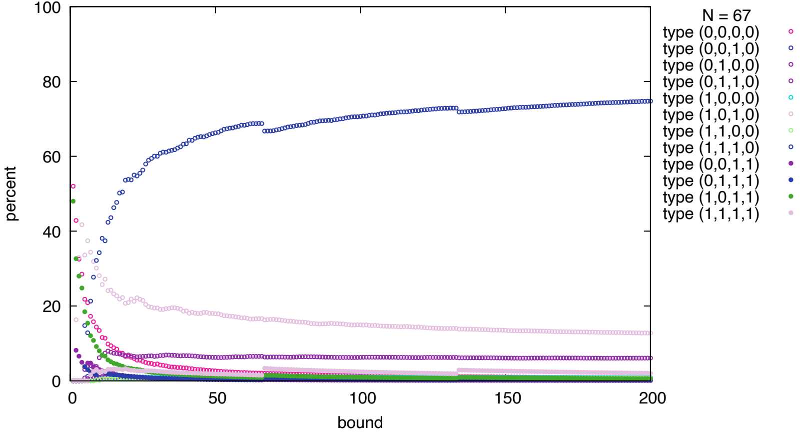


Four eigenspaces $V = V_1 \oplus V_2 \oplus V_3 \oplus E$: $N \in \{67, 73, 89\}$

Table: Lattice indices, and the quotient $Q = (N-1)/\Pi_{i=1}^{3}[\Lambda_i : \Lambda_i']$. $\dim(V_1) = 1$

N	$\dim(V_2)$	$\dim(V_3)$	$[\Lambda_1 : \Lambda_1']$	$[\Lambda_2 : \Lambda_2']$	$[\Lambda_3 : \Lambda_3']$	Q
67	2	2	1	1	11	$2 \cdot 3$
73	2	2	1	1	3	$2^3 \cdot 3$
89	1	5	1	1	11	2^3

Four eigenspaces $V = V_1 \oplus V_2 \oplus V_3 \oplus E$: $N \in \{67, 73, 89\}$



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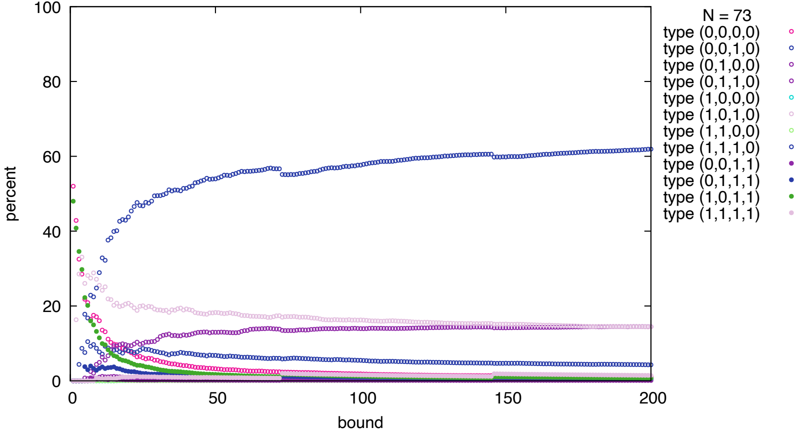

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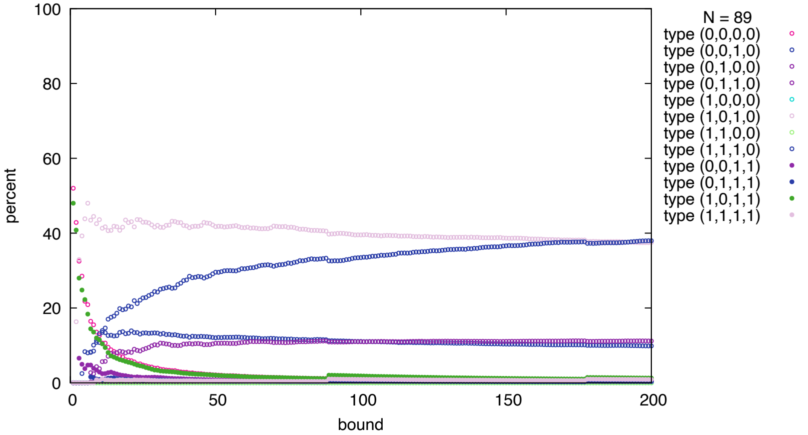


Other directions

- Bianchi modular symbol (GL_2 over imaginary quadratic fields)
- Steinberg homology and group cohomology
- Higher rank modular symbols ($GL_3(\mathbb{Z})$)
- Sharbly complex (GL_2 over CM quartic fields, GL_3 over imaginary quadratic fields)
- Difficulties: analogue of unimodular symbols, reduction algorithm, computationally expensive

Higher rank modular symbols

Theorem (Ash-Rudolph, 1979)

1. *As abelian group,* $\mathrm{St}(\mathbb{Q}^3;\mathbb{Z})$ *is generated by* $[v_1, v_2, v_3]$ *as* v_1, v_2, v_3 *range over all elements of* $\mathbb{Q}^3$*.*
2. *The following relations hold:*
   1. $[v_1, v_2, v_3] = 0$ *if* v_1, v_2, v_3 *do not span* $\mathbb{Q}^3$*.*
   2. $[v_1, v_2, v_3] = [kv_1, v_2, v_3]$ *for any nonzero* $k \in \mathbb{Q}$*;*
   3. $[v_1, v_2, v_3] = (-1)^s[v_{s(1)}, v_{s(2)}, v_{s(3)}]$ *for any permutation* $s \in S_n$*;*
   4. $[v_1, v_2, v_3] = [x, v_2, v_3] + [v_1, x, v_3] + [v_1, v_2, x]$ *for any nonzero* $x \in \mathbb{Q}^n$*.*

We call the fourth relation “passing through x”.

Reduction Algorithm for $n = 3$

[van Geemen-van der Kallen-Top-Verberkmoes 1997]

Let $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, $\det(A) > 1$. There is $m > 1$ and a nonzero vector in the kernel of A modulo m, so there exists $a_1, a_2, a_3 \in \mathbb{Z}$ such that

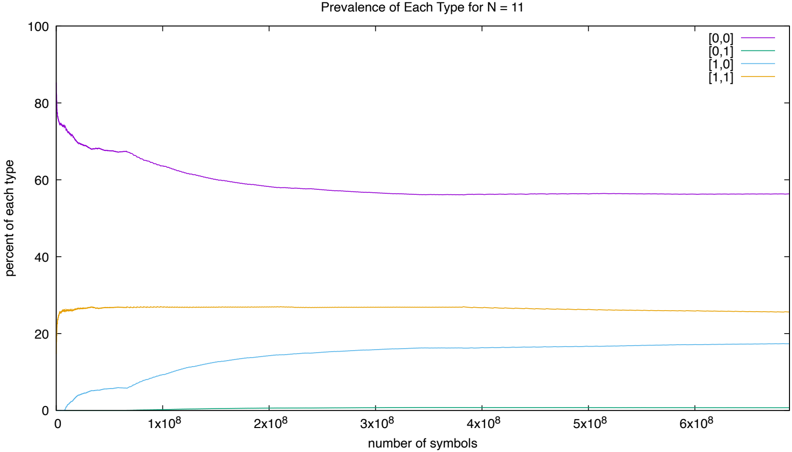
- $x = \frac{1}{m}(a_1 v_1 + a_2 v_2 + a_3 v_3) \in \mathbb{Z}^3$
- $|a_i| \le m/2$

Passing through x shrinks the determinant by a factor of at least $1/2$.

Some preliminary plots

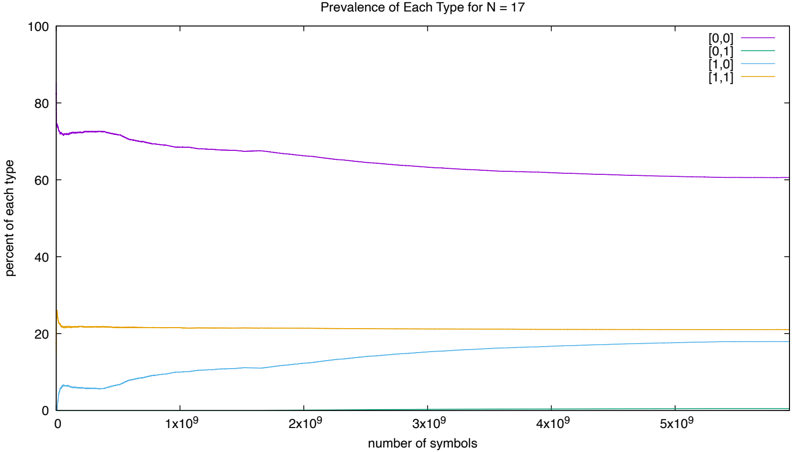
The computations are much more expensive, and it looks like we need to go far to see different phenomena.

$n = 3, N = 11$ plots

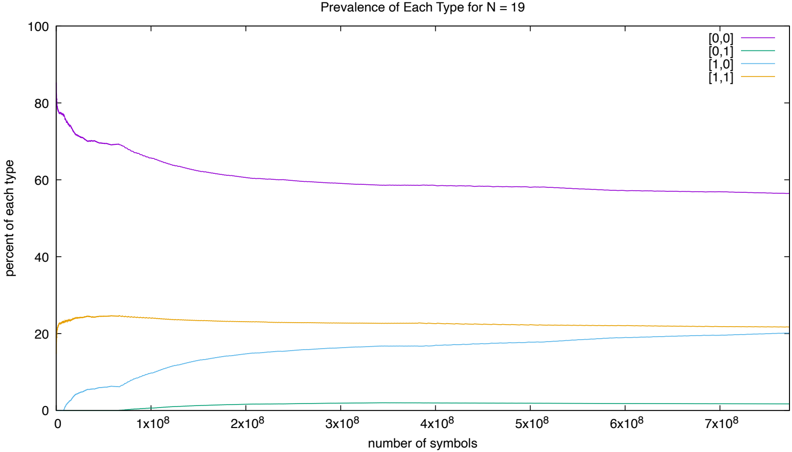


Prevalence of Each Type for N = 11
percent of each type
number of symbols
100
80
60
40
20
0
0
1x10^8
2x10^8
3x10^8
4x10^8
5x10^8
6x10^8
[0,0]
[0,1]
[1,0]
[1,1]

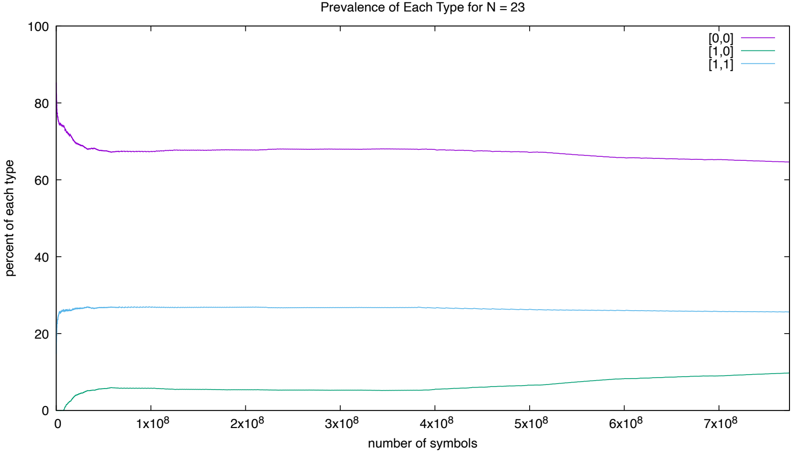
$n = 3$, $N = 17$ plots


Prevalence of Each Type for N = 17
percent of each type
number of symbols
0
20
40
60
80
100
0
1x10^9
2x10^9
3x10^9
4x10^9
5x10^9
[0,0]
[0,1]
[1,0]
[1,1]

$n = 3$, $N = 19$ plots


Prevalence of Each Type for N = 19
100
80
60
40
20
0
percent of each type
0
1x10^8
2x10^8
3x10^8
4x10^8
5x10^8
6x10^8
7x10^8
number of symbols
[0,0]
[0,1]
[1,0]
[1,1]

$n = 3$, $N = 23$ plots


Prevalence of Each Type for N = 23
[0,0]
[1,0]
[1,1]
percent of each type
100
80
60
40
20
0
0
1x10^8
2x10^8
3x10^8
4x10^8
5x10^8
6x10^8
7x10^8
number of symbols

Thank you.