Random Modular Symbols

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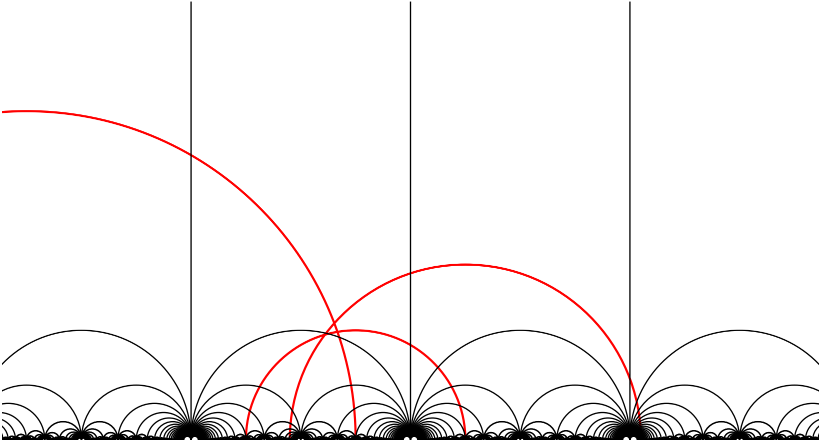
Goal

Understand the image of the set of modular symbols



Modular symbols for $\mathrm{GL}_2(\mathbb{Z})$

As abelian group, $\mathrm{St}(\mathbb{Q}^2;\mathbb{Z})$ is generated by symbols $[v, w]$ as v, w range over all elements of $\mathbb{Q}^2$. The following relations hold:

1. $[v, w] = 0$ if v, w do not span $\mathbb{Q}^2$;
2. $[v, w] = [kv, w]$ for any nonzero $k \in \mathbb{Q}$;
3. $[v, w] = -[w, v]$;
4. $[v, w] = [v, x] + [x, w]$ for any nonzero $x \in \mathbb{Q}^n$.

$$\Gamma = \Gamma_0^{\pm}(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}$$

$$V = H_0(\Gamma, \mathrm{St}(\mathbb{Q}^2;\mathbb{Z})) \otimes \mathbb{Q} \simeq H^1(\Gamma;\mathbb{Q})$$

The co-invariants of the Steinberg module $\mathrm{St}(\mathbb{Q}^2;\mathbb{Z})$ (tensored with $\mathbb{C}$) compute weight 2 modular forms for Γ.

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Types of modular symbols and cuspidal symbols

$$U = \text{image of modular symbols} \subseteq V$$
$$U' = \text{image of cuspidal symbols} \subseteq U$$

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus E, \quad \text{with} \quad \pi_i \colon V \to V_i.$$

Definition

For $v \in V$, define the *type of v* to be

$$t(v) = (t_1, t_2, \ldots, t_k, t_E),$$

where

$$t_\alpha = \begin{cases} 1 & \text{if } \pi_\alpha(v) \neq 0, \\ 0 & \text{if } \pi_\alpha(v) = 0. \end{cases}$$

We examined in detail the types for prime $N < 100$ for 6,496,360,000 modular symbols.

One eigenspace $V = E$: $N \in \{2, 3, 5, 7, 13\}$

U has finite image

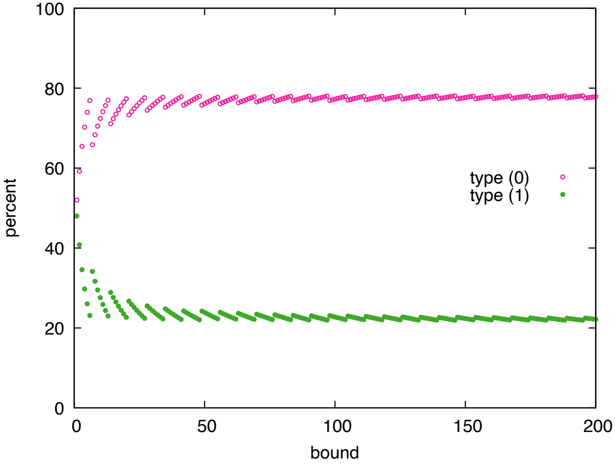


Figure: $N = 7$

Cuspidal $\frac{1+N^2}{(1+N)^2}$ versus non-cuspidal $\frac{2N}{(1+N)^2}$

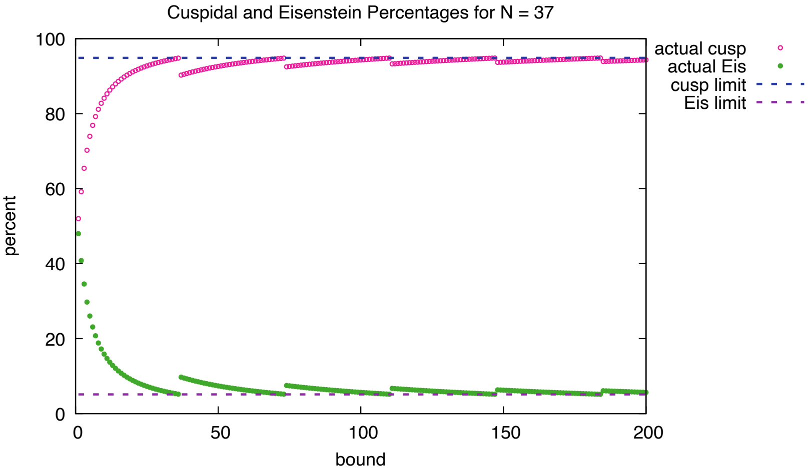


Figure: The percentage of cuspidal and noncuspidal (Eis) modular symbols observed for level 37 as a function of box size.

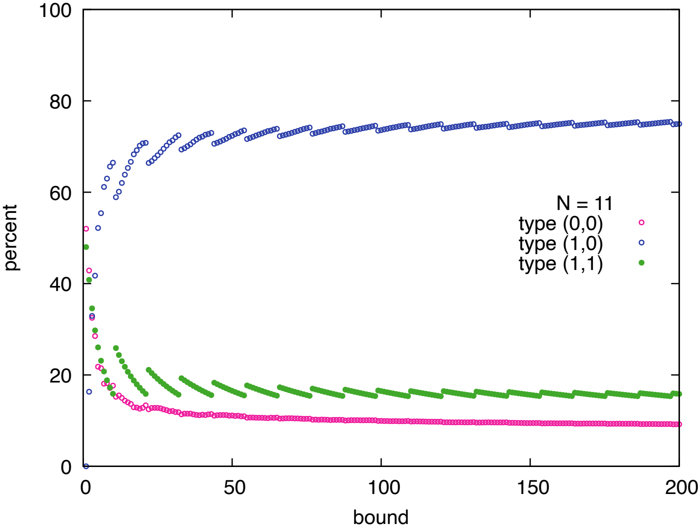
What do U and U' look like?

Let $\Lambda \subset V$ be the $\mathbb{Z}$-lattice generated by U, and define Λ' analogously.

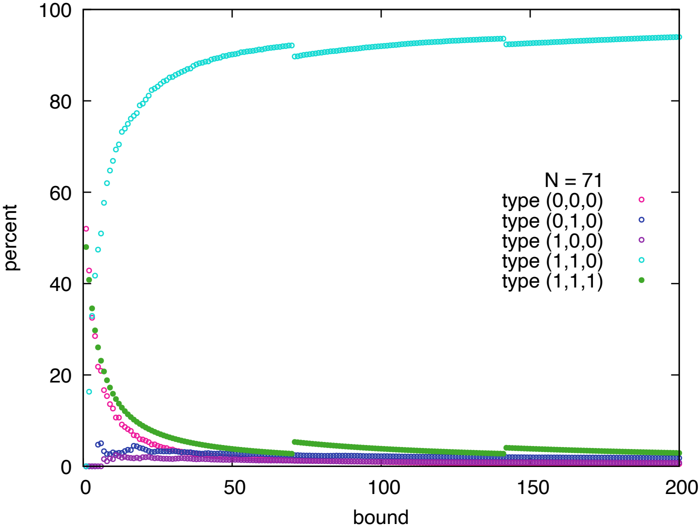
Theorem

1. $U' = \Lambda'$
2. $U = \Lambda' \cup ([e,f]_\Gamma + \Lambda') \cup (-[e,f]_\Gamma + \Lambda')$

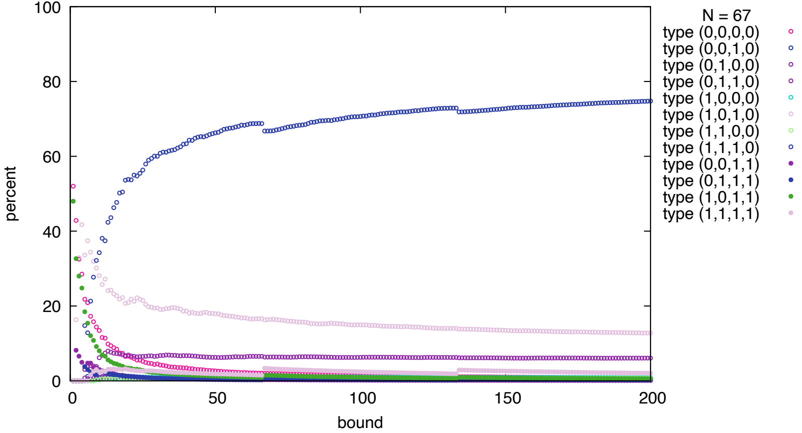
Two eigenspaces $V = V_1 \oplus E$: $N \in \{11, 17, 19, 23, 29, 31, 41, 47, 59\}$



Three eigenspaces $V = V_1 \oplus V_2 \oplus E$: $N \in \{37, 43, 53, 61, 71, 79, 83, 97\}$



Four eigenspaces $V = V_1 \oplus V_2 \oplus V_3 \oplus E$: $N \in \{67, 73, 89\}$



Eisenstein obstruction and congruence of forms

Let $\Lambda_i = \pi_i(\Lambda)$, and define Λ'_i analogously.

Theorem (Eisenstein obstruction)

Suppose $[\Lambda_i : \Lambda'_i] \neq 1$, and let $[v, w]_\Gamma \in V$ have type $(t_1, t_2, \ldots, t_k, t_E)$. Then $t_E = 1$ implies that $t_i = 1$.

Corollary (Congruence of forms)

If prime $\ell > 2$ divides the index $[\Lambda_i : \Lambda'_i]$, then there is a newform of level N and weight 2 whose Hecke eigenvalue a_p is congruent modulo ℓ to $p + 1$ for all $p \neq N$. Such a prime p divides $N - 1$.

$$V = V_1 \oplus E, \quad \dim(V_1) = 1, \quad [\Lambda_1 : \Lambda_1'] = 5$$

There is a newform with LMFDB label 11.2.a.a.

p	2	3	5	7	11	13	17	23	...
a_p	-2	-1	1	-2	1	4	-2	-1	...
$a_p \bmod 5$	3	4	1	3	1	4	3	4	...

Two congruences for $N = 71$

$$V = V_1 \oplus V_2 \oplus E, \quad \dim(V_i) = 3, \quad [\Lambda_1 : \Lambda_1'] = 7, \quad [\Lambda_2 : \Lambda_2'] = 5$$

There are newforms with LMFDB labels 71.2.a.a and 71.2.a.b.
Let α be root of $x^3 - x^2 - 4x + 3$.

p	a_p	$a_p \bmod \mathfrak{p}_7$	b_p	$b_p \bmod \mathfrak{p}_5$
2	$-\alpha$	3	$\alpha^2 - 3$	3
3	α	4	$\alpha^2 - \alpha - 3$	4
5	$-\alpha^2 - \alpha + 5$	6	$-\alpha^2 + 2$	1
7	2α	1	2α	3
11	$2\alpha^2 - 3$	5	-2α	2
13	$-2\alpha^2 + 4$	0	4	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Other directions

- Bianchi modular symbol (GL_2 over imaginary quadratic fields)
- Steinberg homology and group cohomology
- Higher rank modular symbols ($GL_3(\mathbb{Z})$)
- Sharbly complex (GL_2 over CM quartic fields, GL_3 over imaginary quadratic fields)
- Difficulties: analogue of unimodular symbols, reduction algorithm, computationally expensive

Thank you.