MAT 311 Guidelines for Writing Assignments

You will be required to turn in four writing projects during this course. Significant part of your grade on these assignments will be based upon the quality of your writing. Over the years, the mathematical community has agreed upon a number of more-or-less standard conventions for proof writing. These notes are intended to give you some guidance on important matters of style of mathematical exposition.

1. Why should you have to write papers in a math class?

For most of your life so far, the only kind of writing you have done in math classes has been on homework and tests, and for most of your life you have explained your work to people that know more mathematics than you do (that is, to your teachers). But soon, this will change. Professional mathematicians spend most of their time writing: communicating with colleagues, applying for grants, publishing papers, writing memos, syllabi, and course materials (like this one). Writing well is extremely important to mathematicians, since poor writers have a hard time getting published, getting attention from the Deans, and obtaining funding. It is ironic but true that most mathematicians spend more time writing than they spend doing mathematics. But most of all, one of the simplest reasons for writing in a math class is that writing helps you learn mathematics better. By explaining a difficult concept to other people, you end up explaining it to yourself.

2. How is mathematical writing different from what you have done so far?

Good mathematical writing is a skill that takes a good deal of practice and patience to master, just as any other advanced skill. A good mathematical essay has a fairly standard format. We tend to start solving a problem by first explaining what the problem is, often trying to convince others that it is an interesting or worthwhile problem to solve. After stating what the problem is, we usually then state the answer, even before we show how we got it. Sometimes we even state the answer right along with the problem. Explaining the solution and then the answer is usually reserved for cases where the solution technique is even more interesting than the answer, or when the writers want to leave the readers in suspense. But if the solution is messy or boring, then it is typically best to hook the readers with the answer before they get bogged down in details.

Good mathematical papers are the ones that are the easiest to read: clear explanations, uncluttered expositions on the page, well-organized presentation. For example, you *could* say:

> In point of fact, we devolved upon the decision to solicit opinions, form an enumeration, and produce a tally.

But why not instead say:

> We decided to take a vote.

The second sentence says in six words what the first said in 19, and it presents the message more clearly. Mathematics is difficult to read under the best of circumstances. Do not make the reader's job even more challenging by weighing down your prose with excess baggage. Unlike humanities students, mathematicians do not have to worry about over-using "trite" phrases in mathematics.

Most authors avoid using the word "I" in mathematical writing. It is standard practice to use "we" whenever it can reasonably be interpreted as referring to "the writer and the reader." For example,

> We will prove the theorem by induction on n.

But if you are really referring only to yourself, it is better to go ahead and use "I" so you don't sound like the Queen of England:

> I learned this technique from John Doe.

3. Audience

Every writer should have a clear concept of the intended audience for a piece of writing. Only then can the writer be sure that he or she is giving the right amount of information to effectively communicate. This is especially the case for mathematical writing. When mathematicians write to each other, they write in a very brief style with many details omitted. When they write for students, they include more detail. *For the writing assignments which you will be doing, you should take your audience to be your peers in the abstract algebra class.* Write your solutions with the goal of communicating with one of your peers. You may of course assume that they are familiar with terms and theorems from the text, so you can cite it freely.

Since you are a member of the class, you may find it helpful to regard yourself as the audience. Indeed, writing mathematical ideas is an effective means of communicating with yourself. The discipline of putting the ideas down on paper in a logical sequence forces you to think them through clearly.

4. Statement of the problem

If you are writing a proof, you should always precede it with a precise statement, in one or more English sentences, of the result you are proving. Thus, you should begin your writing assignment with a carefully worded statement of the result you are about to establish. *Do not simply repeat the problem the way it is stated in the assignment.* In the text, the problems are usually stated in the imperative mode, using phrases such as "show that …" or "prove that …" You should rework this into a declarative statement of what it is that you will be proving. You may modify the statement to emphasize aspects which you feel need particular attention, or to reflect any generalizations of the original problem which you are making. For example, the original problem is:

> If G is a group and H, K are two subgroups of finite index in G, prove that $H \cap K$ is of finite index in G. Can you find an upper bound for the index of $H \cap K$ in G?

If you were writing a solution to this problem, you might begin as follows:

> **Proposition.** Let H and K be two subgroups of finite index in a group G. Then the intersection $H \cap K$ is a subgroup of G which also has finite index. Moreover,
>
> $$[G : H \cap K] \leqslant [G : H][G : K]. \tag{1}$$
>
> *Proof.* We observe that it is enough to establish (1). Indeed, (1) implies if both $[G : H]$ and $[G : K]$ are finite, then so is $[G : H \cap K]$. (Now you should explain how (1) is derived.)

Note that our restatement of the problem emphasizes the fact that the intersection of two subgroups is a subgroup (otherwise one cannot talk about the index) and it also states an upper bound for the index of the intersection we were asked to find. Notice also that the formula is displayed on a separate line and assigned a number. This makes it very convenient to refer to this formula later. The proof is clearly separated from the statement of the proposition by a blank line and the word "Proof." *A common mistake students make is to run the statement of the claim and its proof together so one cannot tell where the first ends and the second begins.*

Do you see a way to improve our statement of the proposition? (Pause for a minute and think about it.) The hint is in the proof: the fact that the intersection $H \cap K$ has finite index in G follows from the formula (1) and needs no additional proof once we establish (1). Our second, improved, attempt might look like this:

> **Proposition.** Let H and K be two subgroups of finite index in a group G. Then
>
> $$[G : H \cap K] \leqslant [G : H][G : K]. \tag{2}$$
>
> Consequently, $H \cap K$ is a subgroup of G of finite index.
>
> *Proof.* (Now it is enough to explain how (2) is derived.)

5. Abbreviations

There are many abbreviations that we use frequently in informal mathematical communication: "s.t." (such that), "w.r.t." (with respect to), and "w.l.o.g." (without loss of generality) are some of the most common. *These are indispensable for writing on the blackboard and taking notes, but should* ***never*** *be used in written mathematical exposition.* The only exceptions are abbreviations that would be acceptable in any formal writing, such as "i.e." (*id est,* which means "that is") or "e.g." (*exempli gratia,* which means "for example"); but if you use these, be sure you know the difference between them.

One abbreviation that deserves special mention is "iff" (if and only if). Some mathematical writers use this routinely, even in quite formal writing. I suggest that you reserve it for the blackboard and your lecture notes.

6. Mathematical formulas and grammar

Writing about mathematics presents some unique grammatical issues, among them, the usage of mathematical formulas within the text of a sentence. In this document, the word "formula" refers to any expression made up of one or more mathematical symbols, where a "symbol" can be a variable name such as x, y, P, Q, α, β; a function name such as f, sin, log; or any of the special-purpose mathematical symbols that we use to refer to operators and relations such as $+$, $=$, $\in$. There are two kinds of formulas distinguished in mathematical writing:

- Single symbols and most short simple formulas should be included directly within your paragraphs, as in the sentence

  If x is a real number, then $x^2 \geqslant 0$.

  These are called *in-line formulas.*

- Any formula that is large or especially important should be centered on a line by itself; this is called a *displayed formula.* Here is how a displayed formula looks:

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

  If you wish to give a number to a formula in order to refer to it later, the formula must be displayed. Your formula numbers can be placed either at the right margin or at the left margin, as long as you are consistent. If a displayed formula ends a sentence, it must be followed by a period.

Every mathematical symbol or formula, whether in-line or displayed, must have a definite grammatical function as *part of a sentence;* a formula cannot stand on its own as an entire sentence. Formulas should almost always have one of the following two grammatical functions:

- A formula representing a particular mathematical object can be used as a noun;
- a complete symbolic mathematical statement can be used as a clause.

For example, consider the following sentence:

> If $x > 2$, we see that $x^2 + x$ must be greater than 6.

Here "$x > 2$" is a mathematical statement functioning as a clause (whose verb is "$>$"), while "x^2+x" and "6" are mathematical expressions (representing real numbers) that function as nouns. *One useful way to check that your sentences containing formulas are grammatically correct is to read each sentence aloud.* When you do so, bear in mind that many symbols can be read in several different ways — for example, the symbol "$=$" can be read as "equal," "equals," "equal to," "be equal to," or "is equal to," depending on context. An example of good sentence structure is

> Since $A < B$ we know that …

Notice that the sentence reads well aloud: "Since A is less than B we know that …"

A common misuse of mathematical notation is to put it at the beginning of a sentence, because that makes it harder for the reader to recognize that a new sentence has begun. (You cannot capitalize a mathematical symbol to indicate the beginning of a sentence!) It is usually easy to avoid this by minor rewording, for example,

> Let G be a group. G is said to be *abelian* if …

can be rephrased as:

> A group G is said to be *abelian* if …

Short mathematical statements in bulleted or numbered lists, however, can begin with symbols if they are clearer that way.

You should avoid writing two formulas separated only by a comma or other punctuation mark, because they will look like one long formula. For example, the sentence

> If $x \neq 0$, $x^2 > 0$

can be confusing; it would be easier to read if a word were interposed between the two formulas, as in

> If $x \neq 0$, then $x^2 > 0$.

Symbols representing mathematical relations (like $=$, $\leqslant$, $\in$, or $\subseteq$) or operators (like $+$, $-$, or $\cap$) should be used only to connect mathematical formulas, not to connect words with symbols or with each other. For example, do not write:

> If x is a real number that is > 2, then $x^2 + x$ must be > 6. **(BAD)**

Instead, write:

> If x is a real number greater than 2, then $x^2 + x$ must be greater than 6.

Symbols for logical terms, such as $\exists$ (there exists), $\forall$ (for all), $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\Rightarrow$ (implies), and $\iff$ (if and only if), should *never* be used to replace the corresponding words in an English sentence. The only time these symbols have any place in formal mathematical writing is as part of complete symbolic logic formulas. In fact, unless the subject you are writing about is mathematical logic, it is usually clearer to write out the statements in English.

7. Quantification

Quantification is one of the key concepts which separates abstract algebra from your earlier experiences with algebra. This term refers to the specification of the scope of variable by means of the quantifying phrases "for all" and "there exists" or their linguistic equivalents. When we introduce a variable symbol to represent some object, we always want to make it clear whether the symbol can be taken to be any of the objects in the domain, or if we are asserting by its introduction the existence of some object which has particular properties.

For example, the associativity axiom involves the equation

$$(xy)z = x(yz).$$

A critical feature of the axiom is that this equation is understood to be true *for all* of the elements x, y, and z under consideration (for instance, elements of a group). On the other hand, the identity axiom for groups involves the equations

$$ex = x \quad \text{and} \quad xe = x.$$

Here, the assertion is that there is a *particular* element e in the group such that *for every* element x in the group, these equations are true.

You should also be aware of the convention that variables which appear without quantification are understood to be universal. For example, if you read a statement such as

> Since $xy = yx$, we know that $(xy)^n = x^n y^n$,

and there has been no quantification indicated for x and y, you are to understand that the claim is made for all x and y in the domain of discourse.

8. Symbolic manipulations

Any serious writing about algebra must often make use of extensive algebraic manipulations. It takes some practice to develop skill at writing symbolic manipulations which effectively communicate with the reader. An important point to remember in this regard is that the manipulative display which best communicates with the reader is usually not the first one you found when you proved the point for yourself, i.e., if what you write up is simply a chronology of your discovery process, your proof is most likely not going to be as clear to the reader as it would be if you spent some time refining your approach with an eye toward good exposition.

Suppose that you are proving that in a group G, if two elements a and b commute, then a commutes with b^{-1} as well. That is, if $ab = ba$, then $ab^{-1} = b^{-1}a$. You may begin your proof by looking at the equation which you want to prove, and doing some manipulations with it. You might notice that if you multiply both sides on the right by b, then the following happens:

$$\begin{aligned}(ab^{-1})b &= (b^{-1}a)b\\ a(b^{-1}b) &= b^{-1}(ab)\\ a &= b^{-1}(ab).\end{aligned}$$

If you multiply each side of the last equation by b on the left, you get

$$\begin{aligned}ba &= b(b^{-1}(ab))\\ ba &= (bb^{-1})(ab)\\ ba &= ab.\end{aligned}$$

Thus, if the target equation $ab^{-1} = b^{-1}a$ is true, so is the hypothesis $ab = ba$. *This is the converse of what you wanted to prove.* You could simply reverse this development and get a valid proof.

One disadvantage of displays of the sort above is that the reader must study both sides of each equation to see what changes have been made and then consider why these changes are justified. A better approach is to take the mathematics which you have developed during your discovery process and rework it into a single sequence of equalities. For example, we may produce the following proof:

$$
\begin{aligned}
ab^{-1} &= e(ab^{-1}) && \text{Identity axiom} \\
&= (b^{-1}b)(ab^{-1}) && \text{Inverse axiom} \\
&= b^{-1}(ba)b^{-1} && \text{Generalized associative law} \\
&= b^{-1}(ab)b^{-1} && \text{Hypothesis: } ab = ba \\
&= (b^{-1}a)(bb^{-1}) && \text{Generalized associative law} \\
&= (b^{-1}a)e && \text{Inverse axiom} \\
&= b^{-1}a. && \text{Identity axiom}
\end{aligned}
$$

This display allows the reader to focus at each step on a few easily found changes from the previous line. The brief explanation at the right gives the justification of each step. (This is the one circumstance in which sentence fragments will be acceptable.)

Another aspect of good usage of symbolic manipulations has to do with how they are introduced and how they are interpreted. It is not a good idea to write down a series of symbolic manipulations without any introduction of what is going on in English. The introduction must be there to address such issues as the quantification. Also, after the display, you want to give a sentence or two which state the significance of the manipulation for the question under consideration.

Keep in mind that from now on your calculus technique of starting with what you want to prove and working with that expression until everything "checks" is not acceptable.

9. Proofreading

Proofreading is an essential part of the writing process, and it is not a trivial one. You do not simply write the words and then quickly scan them to be sure that there are no gross errors. A prominent mathematics expositor Paul Halmos says that he never publishes a word before he has read it six times. Another good mathematics expositor Steven G. Krantz suggests [3] the following practice:

- One proofreading should be to check *spelling* and simple syntax errors.
- One proofreading should be for *accuracy*.
- One proofreading should be for *organization* and for *logic*.
- One proofreading should be for *sense*, and for the flow of the ideas.
- One proofreading should be for *sound* (reading aloud).

10. Acknowledging sources

You are encouraged to work together as you prepare your homework assignments (but not the writing assignments). If you do so, common standards of academic honesty require that you acknowledge the sources of your ideas. This should be done by means of a note at the end of each exercise. These notes should be as explicit as possible about which insights you obtained from what sources. The more specific you are, the better.

11. Grading criteria and procedures

For each writing assignment, you will need to submit a draft version of the proof by a specified deadline. I will carefully read the submitted papers and explicitly identify particular points that need to be improved; naturally your revision should not exclusively be confined to my markings. I will return the "graded" drafts to you along with some comments and suggestions on how to improve the exposition and the logic of the proof. In some instances I will personally talk to you about your draft version. Then you must write up a final version of the proof taking into account my comments on the earlier version, and submit it by another specified deadline. The quality and lucidity of the presentation in the final submitted version of each writing assignment is expected to be commensurable with that of the proofs in the textbook.

For each writing assignment, you will get one point for submitting a draft version of the proof in time and, moreover, it needs to be a serious attempt: a last minute scribble without much effort will not receive any credit. One point will be awarded if the final version is submitted by the deadline and one more point will be awarded if the final version indicates a significant improvement over the draft version. Finally, one or two extra points will be awarded only if the final version is mathematically correct, i.e., there are no errors or gaps in the proof, and the presentation adheres to the general principles of good mathematical writing outlined in this document.

All writing assignments must be typewritten. In order for the student to pass the course, all stages of each writing assignment must be completed. According to UNCG policies, failure to submit either a draft version or a revised version for at least one writing assignment will result in failing the course.

References

[1] A. Crannell, A Guide to Writing in Mathematics Classes, 1994.

[2] A. Sterrett, ed., Using Writing to Teach Mathematics, Mathematical Association of America, 1992.

[3] S.G. Krantz, A Primer of Mathematical Writing, American Mathematical Society, 1998.