$2x, 2y, 2z$ respectively, we have $\alpha+\beta=90°-\gamma$ so that $\cos(\alpha+\beta)=\sin\gamma$ and hence, $\sqrt{(1-x^2)(1-y^2)}=xy+z$. When this is freed of radicals we obtain (1).

This interpretation suggests a generalization of the law of cosines. Dividing the semicircle of radius 1 into n angles $2\alpha_1, 2\alpha_2, \cdots, 2\alpha_n$, and designating the opposite chords by $2x_1, 2x_2, \cdots, 2x_n$, we have immediately $\cos(\alpha_1+\alpha_2 \cdots +\alpha_n) =0$. The resulting equation when freed of radicals will be symmetric in $x_1, x_2, \cdots, x_n$.

However, even for $n=4$ the equation will be of the sixteenth degree in x_1, x_2, x_3 and x_4. In fact if $\alpha+\beta+\gamma+\delta=90°$ we see that $\cos(\alpha+\beta)=\sin(\gamma+\delta)$.

Hence

$$\sqrt{(1-x^2)(1-y^2)} - u\sqrt{1-z^2} = xy + z\sqrt{1-u^2},$$

where we have called the lengths of the opposite chords $2x, 2y, 2z, 2u$. When this equation is freed of radicals we obtain

$$[(1-x^2-y^2-z^2-u^2)^2-4(x^2y^2z^2+x^2y^2u^2+x^2z^2u^2+y^2z^2u^2-2x^2y^2z^2u^2)]^2$$
$$=64x^2y^2z^2u^2(1-x^2)(1-y^2)(1-z^2)(1-u^2).$$

In terms of the elementary symmetric functions of x^2, y^2, z^2, u^2,

$$r=\sum x^2, \qquad s=\sum x^2y^2, \qquad t=\sum x^2y^2z^2, \qquad w=x^2y^2z^2u^2$$

the preceding equation may be written

$$[(1-r)^2-4(t-2w)]^2 = 64w(1-r+s-t+w).$$

It is not clear how one would go about solving such a diophantine equation.

ON THE $\lim_{\theta\to 0}\cos\theta$

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In most texts on calculus, the proof of the fundamental limit equation $\lim_{\theta\to 0}\sin\theta/\theta=1$ involves the assumption that the limit equation $\lim_{\theta\to 0}\cos\theta=1$ is evident. The apparent reason for this supposedly obvious conclusion must have its basis in the fact that $\cos\theta$ is continuous at $\theta=0$. But the notion of limit is a more primitive one than that of continuity. Hence to avoid the possible confusion which may easily arise in the mind of a beginner in calculus, the teacher should at least assert that the $\lim_{\theta\to 0}\cos\theta$ could be proved to be 1 by direct appeal to the definition of limit. Indeed, we could give the following proof, which would simultaneously introduce the student to the analytical method of proof and also prepare an example for the lesson on continuity, in which the instructor shows that $\cos\theta$ is actually continuous at $\theta=0$. Perhaps it could be left to the students to show it is continuous at any value of θ.

Since this proof would be given at an early stage of a calculus course, prior to which inverse trigonometric functions may not have been covered, it will be desirous to avoid such functions, so we shall define $\cos\theta$ by the conventional method employing rectangular coordinates. This will also prove helpful in establishing the continuity at $\theta=0$, as the old triangle definition will not suffice, there being no triangle.

Placing the vertex of θ at the origin of the rectangular system, with its initial side along the positive x-axis, and measuring θ as usual counterclockwise, we define $\cos\theta$ by choosing an arbitrary point other than (0, 0) on the terminal side of θ having coordinates (x, y) and let $\cos\theta=x/\sqrt{x^2+y^2}$. Since $y\to 0$ as $\theta\to 0$, we wish to establish the equation $\lim_{y\to 0} x/\sqrt{x^2+y^2}=1$. We shall restrict θ to the first or fourth quadrants and for convenience we may choose $x=1$.

To fulfill the definition of limit, for any $\epsilon>0$, we must produce a δ_ϵ such that

$$\left|\frac{1}{\sqrt{1+y^2}}-1\right|<\epsilon \quad \text{for} \quad |y|<\delta_\epsilon.$$

We analyze as follows: for any $y\neq 0$

$$0<\frac{1}{\sqrt{1+y^2}}<1 \quad \text{so that} \quad \left|\frac{1}{\sqrt{1+y^2}}-1\right|=1-\frac{1}{\sqrt{1+y^2}}\cdot$$

Hence we wish to have

$$1-\frac{1}{\sqrt{1+y^2}}<\epsilon$$

and for $0<\epsilon<1$ (for $\epsilon\geqq 1$ this inequality obviously holds)

$$1-\epsilon<\frac{1}{\sqrt{1+y^2}}$$

$$1+y^2<(1-\epsilon)^{-2}$$

$$|y|<\sqrt{(1-\epsilon)^{-2}-1}$$

giving the desired δ_ϵ as

$$\sqrt{(1-\epsilon)^{-2}-1}.$$

A DE-CENTRALIZED CENTROID

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My students are always interested to hear about a region in Quadrant I whose centroid is on the x-axis. This is, of course, slightly impossible. Moreover, this particular region is wide at one end, and tapers out to a sharp point, with the centroid at the end of the point. This is getting more impossible. Finally, the height of impossibility is reached with the statement that the sharp point reaches out infinitely far.

The region is that included between $y=1/x$ and $y=0$, from $x=1$ to $x=\infty$.