Squeeze, Pintch or Sandwich Theorem This article has no author or citation because it is found in most Calcuus books.

Let f, g, h be three functions defined on an interval $[a, b]$. If $a < c < b$ and $f(x) \leq g(x) \leq h(x)$ for all $a \leq x \leq b$ and $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} h(x) = L$, then

$$\lim_{x\to c} g(x) = L$$

Proof. (For a diagram of the theorem see the current Calculus book). It is required to show that for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$(*) \quad 0 < |x - c| < \delta \Rightarrow |g(x) - L| < \epsilon$$

So let ϵ be an arbitrary positive number. We are given $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} h(x) = L$. Therefore we are given

(i) there is $\delta_1 > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$ and

(ii) there is $\delta_2 > 0$ such that $0 < |x - c| < \delta \Rightarrow |h(x) - L| < \epsilon$

We are also given

(iii) $\qquad f(x) \leq g(x) \leq h(x)$ for all $a \leq x \leq b$.

Now we can take $\delta = \min\{\delta_1, \delta_2\}$, and verify (*):

Let $0 < |x - c| < \delta$, since δ is the smaller of δ_1, δ_2, both (i) and (ii) hold, hence $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$. These are equivalent to

$L - \epsilon < f(x) < L + \epsilon$ and $L - \epsilon < h(x) < L + \epsilon$. Thus

$L - \epsilon \leq f(x) \leq g(x) \leq h(x) \leq L + \epsilon$ (using parts of (i), (ii) and (iii))which is the same as saying

$$|g(x) - L| < \epsilon$$

This completes the proof.

Exercise: Application of the Squeeze Theorem.

(a) Explain using graphs why $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist [Hint: First graph values of x of the form $\frac{1}{\pi/2}, \frac{1}{4\pi/2}, \cdot \frac{1}{6\pi/2} \cdots \frac{1}{(2n\pi)/2} \cdots$, and then graph values of x of the form $\frac{1}{3\pi/2}, \frac{1}{5\pi/2}, \cdot \frac{1}{7\pi/2} \cdots \frac{1}{(2n+1\pi)/2} \cdots$

(b) Explain why $\lim_{x\to 0} x \sin \frac{1}{x}$ cannot be found using the product rule for limits. [Hint: The product rule for limits says

$$\lim_{x\to a} (f(x) \cdot g(x)) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x))$$

provided both limits on the right hand side exist

(c) Find $\lim_{x\to 0} x \sin(\frac{1}{x})$.

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