NOTES AND QUESTIONS FOR PERFECT PAIRINGS

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Abstract. Notes and questions about perfect pairings. This arose in the context of a summer reading course from Stein's [1].

Let R be field, and let M, N, and L be vector spaces over R. (We will usually consider $R = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{F}_p$. Our vector spaces will usually be finite-dimensional.) Many of the things below are true even when R is a ring in the context of R-modules.

Exercise 1. $\mathrm{Hom}_R(M, R)$ is the space of linear functionals on M. It is often denoted M^*, and called the *dual space of* M. More generally, let $\mathrm{Hom}_R(M, N)$ denote the set of R-linear maps from M to N. Prove $\mathrm{Hom}_R(M, N)$ is a vector space. Assume that M and N are finite dimensional. Compute the dimension of $\mathrm{Hom}_R(M, N)$.

Definition 1. A R-bilinear map $\langle \cdot, \cdot \rangle \colon M \times N \to L$ is called a *pairing*.

Exercise 2. A good example to keep in mind is the pairing between M^* and M. Specifically, define $\langle \cdot, \cdot \rangle \colon M^* \times M \to R$ by $\langle f, m \rangle = f(m)$. Prove that this is in fact a pairing.

Exercise 3. Suppose $\langle \cdot, \cdot \rangle \colon M \times N \to L$ is a pairing. We can view $\langle \cdot, \cdot \rangle$ as a R-linear map $\Phi_1 \colon M \to \mathrm{Hom}_R(N, L)$. We can also view $\langle \cdot, \cdot \rangle$ as a R-linear map $\Phi_2 \colon N \to \mathrm{Hom}_R(M, L)$. Explain. (Hint for Φ_1: Given $m \in M$, what is the most natural way to get a map from N to L using what is given?)

Definition 2. A pairing is *non-degenerate* if whenever $\langle m, n \rangle = 0$ for all $n \in N$, then $m = 0$.

Exercise 4. Explain non-degeneracy in terms of Φ_1 or Φ_2.

Definition 3. A pairing is *perfect* if Φ_1 is an isomorphism.

Exercise 5. If $\langle \cdot, \cdot \rangle$ is a perfect pairing of finite-dimensional vectors spaces, is Φ_2 is an isomorphism?

Exercise 6. Is every perfect pairing non-degenerate? Explain.

Exercise 7. Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\mathbb{R}^n$. Prove that $\langle \cdot, \cdot \rangle$ is a non-degenerate, perfect pairing.

Exercise 8. For each pair of vectors u and v in R^2, define $\langle u, v \rangle$ to be the determinant of the matrix with columns u and v. Prove $\langle \cdot, \cdot \rangle$ is a pairing. Is it nondegenerate? Is it perfect?

Exercise 9. For each $A \in \mathrm{Mat}_n(R)$ and each $v \in R^n$, define $\langle A, v \rangle = Av$. Is this a pairing? Is it nondegenerate? Is it perfect?

Exercise 10. For each $f, g \in C^\infty(\mathbb{R})$, define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

Is this a pairing?

Exercise 11. For each $f \in C^\infty(\mathbb{R})$ and each closed interval $[a,b] \subset \mathbb{R}$, define

$$\langle f, [a,b] \rangle = \int_a^b f(x)\, dx.$$

Is this a pairing? Before answering that, think carefully about what you would need to show. What is M, N, and L in this case?

References

[1] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

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