# SOME NOTES FOR SUMMER READING ON MODULAR FORMS

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## 1. Notation

$$
\begin{aligned}
\mathfrak{h} &= \text{complex upper half-plane}\\
G &= \mathrm{SL}_2(\mathbb{R})\\
K &= \mathrm{SO}(2)\\
C &= \text{cone of positive definite symmetric matrices}\\
\Gamma_0(N) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; c \in N\mathbb{Z} \right\}\\
\mathbb{P}^1(\mathbb{Q}) &= \mathbb{Q} \cup \{\infty\} = \text{cusps in } \mathfrak{h}\\
M_k(\Gamma_0(N)) &= \text{weight } k \text{ modular forms for } \Gamma_0(N)\\
S_k(\Gamma_0(N)) &= \text{weight } k \text{ cusp forms for } \Gamma_0(N)\\
\mathbb{M}_2 &= \text{space of modular symbols}\\
\mathbb{M}_2(\Gamma_0(N)) &= \Gamma_0(N)\backslash\mathbb{M}_2\\
&= \text{space of modular symbols for } \Gamma_0(N)
\end{aligned}
$$

## 2. Actions

1. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathfrak{h}^*$,

$$g \cdot z = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \\ \frac{a}{c} & \text{if } z = \infty \end{cases}.$$

Note that if we extend the actions above to an action of $\mathrm{GL}_2^+(\mathbb{R})$, scalar matrices act trivially on $\mathfrak{h}$.

2. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $Q \in \bar{C}$,

$$g \cdot Q = gQg^t.$$

Note that scalar matrices act by homothety (positive scaling) on $\bar{C}$. In particular, scalar matrices act trivially on $\bar{C}/\mathbb{R}_{>0}$.

3. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ and $f \in M_k(\Gamma_0(N))$,

$$f^{[g]_k}(z) = \det(g)^{k-1}(cz+d)^{-k} f(\gamma \cdot z).$$

NOTE: This is a right-action.

4. For $g \in \mathrm{GL}_2(\mathbb{Z})$ and $v \in \mathbb{Z}^2$,

$$g \cdot q(v) = q(gv).$$

WARNING: While the action above is fine when we mod out by homothety and look in $C/\mathbb{R}_{>0}$, many computations we do will be purely in terms of $v$. In that case, always make sure that you scale $\begin{bmatrix} a \\ b \end{bmatrix}$ by $1/\gcd(a,b)$ to ensure that $v$ and $gv$ are primitive.

5. For $g \in \Gamma_0(N)$ and $(c : d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$,

$$(c : d) \cdot g = \text{line in } (\mathbb{Z}/N\mathbb{Z})^2 \text{ through } \begin{bmatrix} c & d \end{bmatrix} g.$$

Note this is a right action.

## 3. Voronoi polyhedron

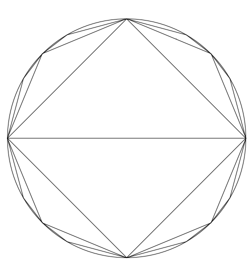
Define a map $q : \mathbb{Z}^2 \to \bar{C}$ by $q(v) = vv^t$. The *Voronoi polyhedron* is the infinite polyhedron $\Pi$ gotten by taking the convex hull

$$\Pi = \mathrm{Convex}\{q(v) \ : \ v \in \mathbb{Z}^2 \setminus 0\}.$$

The vertices of $\Pi$ are the *cusps of* $C$ and are in bijection with the cusps of $\mathfrak{h}$.

## 4. Identifications

$$\begin{aligned}
\mathfrak{h} \simeq C/\mathbb{R}_{>0} \quad & \text{via } z = g \cdot i \to \mathbb{R}_{>0} \cdot gg^t \\
& \text{and } g \cdot i \leftarrow \mathbb{R}_{>0} \cdot gg^t = \mathbb{R} \cdot Q \\
\mathbb{P}^1(\mathbb{Q}) \simeq \text{cusps of } C \quad & \text{via } \frac{a}{b} \text{ (as a reduced fraction)} \to q(\begin{bmatrix} a \\ b \end{bmatrix}), \\
& \quad \infty \to q(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \\
\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \simeq \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z}) \quad & \text{via } (0:1) \cdot g \to \Gamma_0(N) g \\
& \text{and } (c : d) \to \Gamma_0(N) \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\end{aligned}$$



Projection onto trace 1 slice.

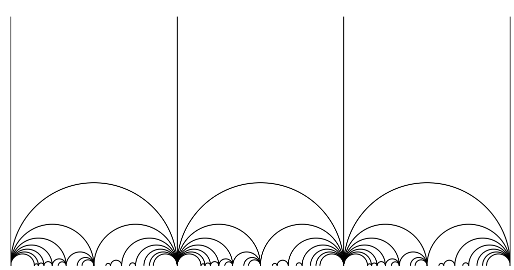


Image under identification with $\mathfrak{h}$.

## 5. Modular symbols

In this section, I will try to be more explicit about the different ways we look at modular symbols. In this section, let $\mathbf{u}_0$ denote the modular symbol $\{0, \infty\}$.

Our goal is to compute the action of Hecke operators $T_p$ with $p$ prime on $\mathbb{M}_2(\Gamma_0(N))$. Then the general theory will tell us that we can recover the action of Hecke operators on modular forms.

The general plan of attack is the following:

1. Lift an element of $\mathbf{u} \in \mathbb{M}_2(\Gamma_0(N))$ to an element of $\tilde{\mathbf{u}} \in \mathbb{M}_2$, expressed as a linear combination of Manin generators.
2. Compute the Hecke action on $\tilde{\mathbf{u}}$. In general, this is no longer a linear combination of Manin generators.
3. Use reduction algorithm (see Homework 4 and Proposition 3.11 (continued fractions) of the textbook) to express in terms of Manin generators.
4. Push down to $\mathbb{M}_2(\Gamma_0(N))$.

Homework 4 deals with 2 and 3, and it should be reviewed for more details.

Manin shows that $\mathbb{M}_2(\Gamma_0(N))$ is generated by $\{r \cdot \mathbf{u}_0\}$, where $r$ ranges over coset representatives of $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Call this the *Manin generators or Manin symbols for* $\Gamma_0(N)$. Note that we may have nontrivial linear relations among the $r \cdot \mathbf{u}_0$s. In particular, $\{r \cdot \mathbf{u}_0\}$ is a spanning set, but may not be linearly independent. By the identifications above, we see that the Manin generators are in bijection with the points in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Indeed if we set `M = ModularSymbols(N,2)` and compute `M.manin_generators()`, you should get the elements of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, which should match what you get if you list the elements of `P1List(N)`.

How far off from $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ is $\mathbb{M}_2(\Gamma_0(N))$? Manin's result Theorem 3.13 of the text gives the answer, but I will try to explain a different way of computing it, so that we can generalize later in the semester.

Let $U \subset \mathbb{M}_2$ denote the set of *unimodular symbols*. These are precisely the $\{\alpha, \beta\}$ with "determinant" $= \pm 1$. They correspond to the edges of the Voronoi polyhedron, or equivalently, the edges of the hyperbolic triangulation shown in the Figure above. Note that one of the consequences of Proposition 3.11 is that $U = \mathrm{SL}_2(\mathbb{Z}) \cdot \mathbf{u}_0$, and we want to compute $\mathbb{M}_2(\Gamma_0(N))$. This is essentially $\Gamma_0(N)\backslash U$ modulo some extra relations.

Let $G_0$ be the stabilizer of $\mathbf{u}_0$ up to sign. In other words,

$$G_0 = \{g \in \mathrm{SL}_2(\mathbb{Z}) \ : \ g \cdot \mathbf{u}_0 = \pm \mathbf{u}_0\}.$$

Then since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $U$, we have that

$$U \simeq \mathrm{SL}_2(\mathbb{Z})/G_0.$$

Note that we want to compute the space $\mathbb{M}_2(\Gamma_0(N)) = \Gamma_0(N)\backslash U$. Therefore we want to compute

$$\begin{aligned} \text{one-cells} &= \Gamma_0(N)\backslash U \\ &= \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})/G_0 \\ &= \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})/G_0. \end{aligned}$$

In other words, the elements of $\mathbb{M}_2(\Gamma_0(N))$ correspond to right $G_0$ orbits in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. See Homework 5.

All that remains is to mod out by the relation

$$\{\alpha, \beta\} + \{\beta, \gamma\} = \{\alpha, \gamma\}.$$

This is essentially the homology relation that goes around the boundary of an ideal triangle. In this context, it suffices to mod out by

$$r \cdot \{0, 1\} + r \cdot \{1, \infty\} + r \cdot \{\infty, 0\}.$$

Translate this relation in terms of projective orbits in Homework 5.

This shows the general plan of attack for computing Hecke operators:

1. Compute $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})/G_0$ modulo triangle relations. This should be the same as the output from `ModularSymbols(N,2)` in SAGE.
2. Lift a basis of 1. to elements of $\mathbb{M}_2$.
3. Compute the Hecke operators on the basis computed in 2.
4. Reduce 3. to get expression in $U$.
5. Express 4. in terms of projective orbits.
6. Mod out by triangle relations.

Homework 4 deals with 3. and 4. Homework 5 deals with 5. and 6.

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