HOMEWORK 4

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We now generalize to consider modular forms for $\Gamma_0(N)$, but restrict our focus to weight 2 forms. Weight 2 forms have a nice description in terms of modular symbols. Read Chapter 3.1-3.4 carefully.

In this homework, we will start to mimic the computations of Hecke operators on weight 2, level N modular forms using modular symbols. We will use a geometric replacement for the continued fractions algorithm for the reduction of modular symbols (Proposition 3.11).

Recall that we identify the cone modulo scaling with the upper half-plane $\mathfrak{h}$. We can view the cone modulo scaling as a disk. Special points on the boundary of the disk will correspond to cusps. Let's recall how this went. The cusps in $\mathfrak{h}$ are $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. We have a map

$$q : \mathbb{Z}^2 \to \bar{C}$$

defined by

$$q(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}.$$

If we view a rational number $\frac{s}{t}$ as a vector $\begin{bmatrix} s \\ t \end{bmatrix}$, then the map q allows us to think of it as a point in $\bar{C}$. Modding out by scalars multiplication, we can view it as a point on the boundary of the disk. The cusp ∞ corresponds to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that under this correspondence, the action of a matrix g on a cusp of $\mathfrak{h}$ translates to regular matrix multiplication on the corresponding vector. Specifically, for $\alpha = \frac{s}{t}$,

$$g \cdot \alpha = \frac{a\alpha + b}{c\alpha + d}$$

and

$$g \cdot q(\begin{bmatrix} s \\ t \end{bmatrix}) = q(g \begin{bmatrix} s \\ t \end{bmatrix}).$$

Now an element $\{\alpha, \beta\} \in \mathbb{M}_2$ can be viewed as a directed line joining two "rational" points on the boundary of the disk. For example, $\{0, \frac{4}{7}\}$ can be thought of in the cone picture as the line joining $q(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ and $q(\begin{bmatrix} 4 \\ 7 \end{bmatrix})$.

The purpose of this exercise is to implement a geometric analogue of the continued fraction algorithm for expressing modular symbol as a sum of *unimodular symbols*. The space of unimodular symbols is spanned by modular symbols $\{\alpha, \beta\}$, where if $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{d}$ then $ad - bc = \pm 1$. For example, the modular symbol $\{0, \frac{4}{7}\}$ is not unimodular since $\det(\begin{bmatrix} 0 & 4 \\ 1 & 7 \end{bmatrix}) \neq \pm 1$. The "traditional method" for expressing $\{0, \frac{4}{7}\}$ as a sum of unimodular symbols uses the theory of continued fractions (see p. 43 example 3.12.) Continued fractions

allows us to see that

$$\{0, \frac{4}{7}\} = \{0, \infty\} + \{\infty, 0\} + \{0, 1\} + \{1, \frac{1}{2}\} + \{\frac{1}{2}, \frac{4}{7}\}.$$

The next few lines in the example, then collect terms modulo $\Gamma_0(11)$. We will worry about collecting terms later, and just worry now about the first part.

In the following, we view modular symbols as spanned by pairs of vectors $[u, v]$ with $u, v \in \mathbb{Z}^2$.

1. Make sure your functions from the last assignment work as expected. Share with peers to be sure to have good code. The deeper we go, the more difficult it will be to see where the bugs are coming from.

2. Write a function `better` which takes as input a pair of vectors $[u, v]$, where $u, v \in \mathbb{Z}^2$ and returns something better. Specifically, if $[u, v]$ is unimodular $(\det(u, v) = \pm 1)$, the code should return a list with just the pair of vectors $[[u, v]]$. If $[u, v]$ is degenerate $(\det(u, v) = 0)$, then return the empty list $[]$. Otherwise, return a pair $[[u, w], [w, v]]$, where $[u, w]$ and $[w, v]$ are "better" than $[u, v]$ in the sense that

$$0 < |\det(u, w)| < |\det(u, v)| \quad \text{and} \quad 0 < |\det(w, v)| < |\det(u, v)|.$$

   (Hint: What are good candidates for w? Geometrically, we can do the following. Compute the barycenter (midpoint) b of the line between $q(u)$ and $q(v)$. Then note that $b \in C$ since $\bar{C}$ is convex. The vertices of the cone containing b are $q(w_1), q(w_2), q(w_3)$ for some vectors $w_1, w_2, w_3 \in \mathbb{Z}^2$. These are good candidates for w. You can compute w_1, w_2, w_3 using the code from Homework 2.)

3. Write a function `reduce` which takes a pair of vectors $[u, v]$ with $u, v \in \mathbb{Z}^2$, and returns a list of pairs $[A_1, \cdots, A_n]$, where $A_i = [u_i, v_i]$ with $u_i, v_i \in \mathbb{Z}^2$ such that
   (a) Each A_i corresponds to a unimodular symbol. (Hint: $\det(u_i, v_i) = \pm 1$.)
   (b) $[A_1, \cdots, A_n]$ represents the same homology class as $[u, v]$. (Hint: This is guaranteed if you arrange that $u_1 = u$, $v_n = v$, and $v_i = u_{i+1}$.)

4. Write a function `hecke_action` which takes as input a prime p, level N, and pair of vectors $[u, v]$, and returns a list of unimodular symbols corresponding to the Hecke translate of $[u, v]$. (Hint: the first part of 3.4.1 gives the list of matrices you need to act by in order to get the Hecke action of T_p. Note that the list depends of whether p divides N or not. Let L be this list. Act on $[u, v]$ by each element of L to get a list of non-unimodular symbols S. Then use your function `reduce` on each entry of S.

5. More problems to come soon...Perhaps Homework 5.

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