HOMEWORK 3

DAN YASAKI

Recall that $\mathfrak{h}$ can be identified with a 2 dimensional submanifold of the open cone C of 2×2 real, positive definite, symmetric matrices. The group $\mathrm{GL}_2(\mathbb{R})$ acts on C by

$$g \cdot Q = aQg^t, \tag{1}$$

where $g \in \mathrm{GL}_2(\mathbb{R})$ and $Q \in C$. Positive reals $H = \mathbb{R}_{>0}$ act on C by scaling (known as *homothety*), and we can identify C/H with $\mathfrak{h}$. Specifically, the point $z \in \mathfrak{h}$ is $g \cdot i$ for some $g \in \mathrm{SL}_2(\mathbb{R})$ (recall Homework 2). Then gg^t is a positive definite symmetric matrix. We identify $\{t(gg^t) \mid t \in H\} \subset C$ with $z \in \mathfrak{h}$.

We have a map $q\colon \mathbb{Z}^2 \to C$ defined by $q(v) = vv^t$. Each element $A \in C$ can be thought of as a binary quadratic form $\phi(v) = vAv^t$. There is an inner product on C given by trace:

$$\langle S, T\rangle = \mathrm{Tr}(ST).$$

This inner product of compatible with q and the form picture of C in the sense that $\langle \phi, q(v)\rangle = \phi(v)$, when suitably interpreted: ϕ on left is matrix in C, ϕ on right is binary quadratic form.

Recall that the Voronoi polyhedron Π is the convex hull of $\{q(v) \mid v \in \mathbb{Z}^2 \setminus 0\}$. By taking cones over the facets of Π, we get a decomposition of C into polyhedral cones. The facets of Π are ideal triangles. To vectors $v, w \in \mathbb{Z}^2$ determine an edge of Π if and only if the determinant of the matrix with columns v, w has determinant ± 1. Each facet F of Π corresponds to a *perfect form* ϕ_F.

In this homework exercise, you will begin to create a Sage package to deal with modular forms in this context. Slowly, we should be able to duplicate many of the procedures for modular forms currently implemented in Sage. We will work in this cone context, so that we can eventually generalize to modular forms over imaginary quadratic number fields.

We identify a triple $[a, b, c]$ of real numbers with the symmetric matrix $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

1. Read `http://www.sagemath.org/doc/developer/conventions.html`, and make sure your code (with comments) is up to snuff.
2. Write a function `cone_to_h`, which takes as input a triple $[a, b, c]$, and returns the point $x + iy \in \mathfrak{h}$ that the corresponds to it.

3. Write a function h_to_cone, which takes as input a point $z \in \mathfrak{h}$, and returns a point in C that corresponds to it.
4. Check your maps above, composing yours and your peer's functions. Specifically, make sure that for several random test values
   (a) `h_to_cone(cone_to_h([a,b,c])) = t[a,b,c]` for some positive real t.
   (b) `cone_to_h(h_to_cone(z)) = z`.
5. Write a function is_conjugate which takes as input 2 triples, A and B, and returns a Boolean representing if there is a matrix $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g \cdot A = B$. [Be sure that your code is easily modifiable to run when/if we change to $\mathrm{GL}_2(\mathbb{Z})$.]
6. Write a function cone_containing which takes as input a triple $[a, b, c]$ and returns a list of vectors $L \subset \mathbb{Z}^2$ which define the polyhedral cone containing $[a, b, c]$.

   [Hint: One approach is to use the fact that modulo the action of $\mathrm{SL}_2(\mathbb{Z})$, there is one facet of Π. It has vertices (minimal vectors) e_1, e_2, and $e_1 + e_2$. In $\mathfrak{h}$, this corresponds to the ideal triangle with vertices 0, 1, and ∞. Find a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ which sends your given point to a point inside this standard cone. Then translate the vertices back to find the cone containing your given point. You can use http://www.math.umass.edu/~gunnells/pubs/msa/msa.ps, Theorem 3. There are also other reduction algorithms you can look for.]
7. Read Chapter 2.4, start reading Chapter 3.

Dan Yasaki, Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27402-6170, USA
E-mail address: d_yasaki@uncg.edu