MAT 191 Test 4 November 18, 2013 (9:00–9:50 AM)

Name: ____________________ Academic Integrity Signature: ____________________

I have abided by the UNCG Academic Integrity Policy.

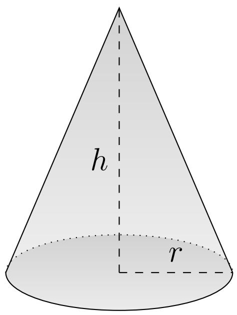
Read all of the following information before starting the exam:

- It is to your advantage to answer ALL of the 11 questions.
- It is your responsibility to make sure that you have all of the problems.
- There is no need to complete the test in order. The problems are independent.
- Correct numerical answers with insufficient justification may receive little or no credit.
- Clearly distinguish your final answer from your scratch work with a box or circle.
- *Budget your time!*
- If you have read all of these instructions, draw a happy face here.

Page:	2	3	4	5	6	7	8	Total
Points:	10	20	20	10	20	10	10	100
Score:								

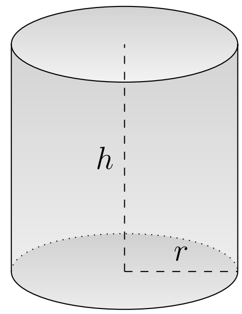
Geometry Formulas

Cone



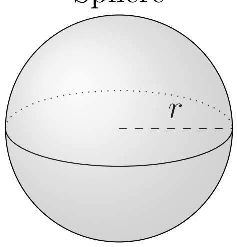
$$V = \frac{1}{3}\pi r^2 h$$
$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$

Cylinder



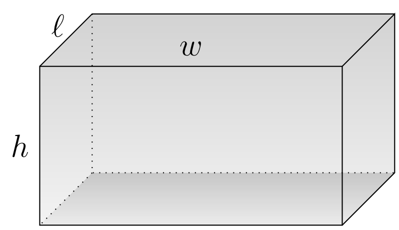
$$V = \pi r^2 h$$
$$S = 2\pi r h + 2\pi r^2$$

Sphere



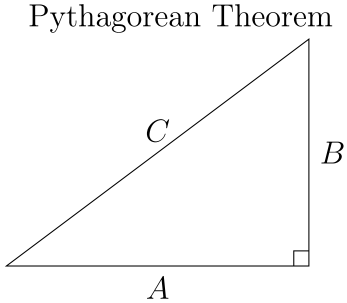
$$V = \frac{4}{3}\pi r^3$$
$$S = 4\pi r^2$$

Rectangular Solid



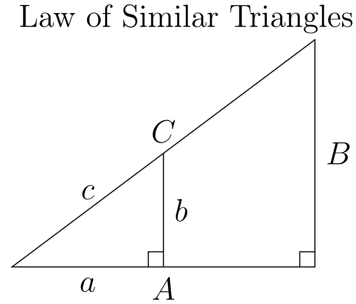
$$V = \ell w h$$
$$S = 2\ell w + 2\ell h + 2hw$$

Pythagorean Theorem



$$A^2 + B^2 = C^2$$

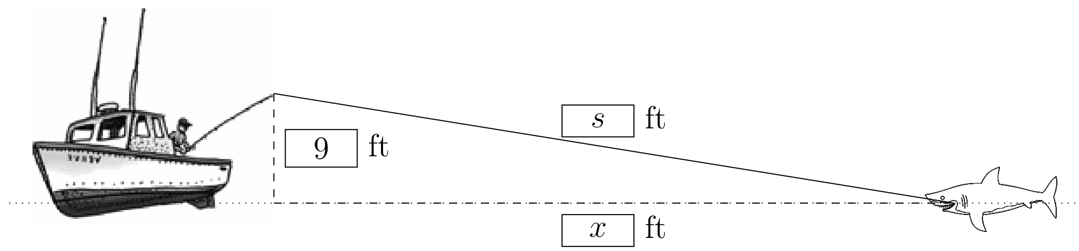
Law of Similar Triangles



$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

1. Billy Bob goes out on a boat for a bit of fishing and hooks a large shark at the surface of the water. He holds the fishing rod steady so that the tip of the rod remains 9 ft above the surface of the water.

   (a) (3 points) Complete the picture below by filling in the boxes with the lengths of the dashed lines and fishing line. Use variables for quantities that change and numbers for quantities that remain the same.



   (b) (7 points) If the shark swims along the surface of the water toward the boat at a speed of $7\frac{\text{ft}}{\text{sec}}$, how fast does Billy Bob in the line in order to keep the line taut when the shark is 40 ft away? (You may use the fact that $\sqrt{1681} = 41$.)

Solution: We have $x^2 + 9^2 = s^2$. Differentiating with respect to time, we get

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}.$$

We are given that $\frac{dx}{dt} = -7$. We want to find $\frac{ds}{dt}$ when $x = 40$. When $x = 40$, we have $9^2 + 40^2 = 1681 = s^2$, so $s = 41$. Plugging in, we get

$$2(40)(-7) = 2(41)\frac{ds}{dt}.$$

Solving, we have $\frac{ds}{dt} = -\frac{280}{41}$. Billy Bob must reel in the line at $\frac{280}{41}\ \frac{\text{ft}}{\text{sec}}$.

2. (5 points) Let f be a differentiable function. Newton's method produces a sequence $x_1, x_2, x_3, \ldots$ of approximate solutions to $f(x) = 0$ given an initial guess x_0. Complete the formula for computing this sequence.

$$\boxed{x_{n+1}} = \boxed{x_n - \frac{f(x_n)}{f'(x_n)}}$$

3. (8 points) For each of the following statements, circle True if the statement must be true and False if it is ever false. No justification is required.

   (a) True | $\boxed{\text{False}}$: If a square grows larger, so that its side length increases at a constant rate, then its area will also increase at a constant rate.

   (b) $\boxed{\text{True}}$ | False: The *differential* of a differentiable function g is $dg = g'(x)dx$.

   (c) True | $\boxed{\text{False}}$: If f and g are differentiable functions, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

   (d) $\boxed{\text{True}}$ | False: If f is a differentiable function, then the *linearization of f at a* is the approximating function $L(x) = f(a) + f'(a)(x - a)$.

4. Let $f(x) = \sqrt[3]{x}$.

   (a) (5 points) Find the linearization $L(x)$ of f at 8.

   Solution: We compute $f(8) = \sqrt[3]{8} = 2$. The derivative is $f'(x) = \frac{1}{3}x^{-2/3}$, so we have

   $$f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{12}.$$

   It follows that the linearization is

   $$L(x) = 2 + \frac{1}{12}(x - 8).$$

   (b) (2 points) Use the linearization to approximate $\sqrt[3]{8.12}$. Simplify.

   Solution: We use the linearization from (a) to get

   $$\begin{aligned} L(8.12) &= 2 + \frac{1}{12}(8.12 - 8) \\ &= 2 + \frac{1}{12}(0.12) \\ &= 2 + 0.01 \\ &= 2.01. \end{aligned}$$

   Note: The actual value of $\sqrt[3]{8.12}$ is $2.0099504125453050240272507 5675\ldots$ so we see that we get very close.

5. (10 points) You measure a cube of gold and find that the edge is 10 cm. Use differentials to estimate the change in volume of a cube if the length is 10.1 cm instead. Include the correct units. Note: I do not want the exact difference.

Solution: Let x be the length of the edge. Then the volume of the cube is $V = x^3$. It follows that the differential is $dV = 3x^2 dx$. Then the approximate change in volume is

$$dV = 3(10^2)(0.1) = 30 \text{ cm}^3.$$

6. (10 points) Suppose that $p(x) = -x^3 + \frac{9}{2}x^2 - 6x$ represents the profit (in millions of dollars) of Link Inc., where x represents millions of swords produced. Find the production level that maximizes profit.

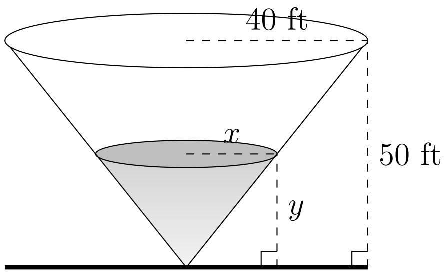
Solution: Note that the domain of p is $0 \leq x < \infty$.

We compute

$$p'(x) = -3x^2 + 9x - 6 = -3(x^2 - 3x + 2) = -3(x-1)(x-2).$$

We see that the critical points are $x = 1, 2$. Making a sign chart, we see that p is decreasing on $(0,1) \cup (2, \infty)$ and increasing on $(1, 2)$. By the First Derivative Test, we have a local maximum when $x = 2$. Since $p(0) = 0$, we see that $p(2)$ is the global maximum. In other words, Link Inc. should produce 2 million swords.

7. (10 points) Uranianite is draining out of the conical cargo hold of an alien spaceship shown below. The vertex is pointing down, the radius is 40 ft, and the height is 50 ft. Suppose the uranianite is draining out at the rate of 5 $\frac{\text{ft}^3}{\text{sec}}$. How fast is the uranianite depth y falling when the uranianite is 25 ft deep? Make sure you include the units of your answer.



Solution: Using similar triangles, we have that $\frac{x}{y} = \frac{40}{50}$. It follows that $x = \frac{4}{5}y$. The volume V of uranianite is given by

$$V = \frac{\pi}{3}x^2y = \frac{\pi}{3}\left(\frac{4}{5}y\right)^2 y = \frac{16\pi}{75}y^3.$$

We differentiate both sides with respect to time using chain rule

$$\frac{dV}{dt} = \frac{16\pi}{75}(3y^2\frac{dy}{dt}) = \frac{16\pi}{25}y^2\frac{dy}{dt}.$$

Now we need to plug in specifics. We are given $\frac{dV}{dt} = -5$ and $y = 25$ at the time of interest. Plugging it all in

$$\begin{aligned}
\frac{dV}{dt} &= \frac{16\pi}{25}y^2\frac{dy}{dt}\\
-5 &= \frac{16\pi}{25}(25^2)\frac{dy}{dt}\\
-5 &= 400\pi\frac{dy}{dt}\\
-\frac{1}{80\pi} &= \frac{dy}{dt}
\end{aligned}$$

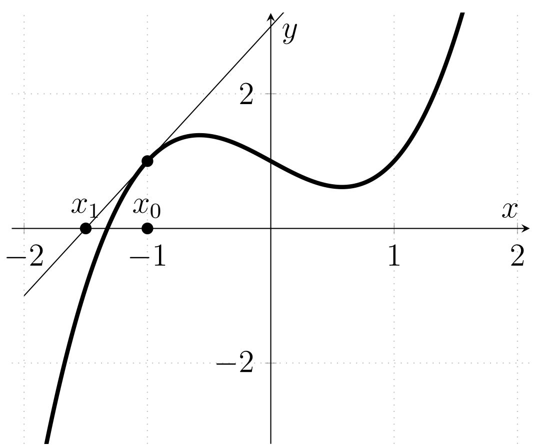
The uranianite depth is falling at $\frac{1}{80\pi}\frac{\text{ft}}{\text{sec}}$.

8. (10 points) Write a checkmark (✓) under each of the indeterminate forms.

$\infty \cdot \infty$	$\frac{\infty}{\infty}$	0^0	∞^∞	$\frac{0}{0}$	1^∞	0^1	$\frac{0}{1}$
	✓	✓		✓	✓		

$\infty - \infty$	$\frac{0}{\infty}$	0^∞	$\infty \cdot 0$	$\frac{\infty}{0}$	∞^0	$\frac{0}{\infty}$	$\frac{1}{\infty}$
✓			✓		✓		

9. (10 points) The graph of $f(x) = x^3 - x + 1$ is shown below. Estimate the real root of f using one iteration of Newton's method with initial guess $x_0 = -1$. i.e. Compute x_1. On the graph, mark the initial point x_0, the next approximation x_1, and the sketch the tangent line to the curve at $x = x_0$.



Solution: We compute $f'(x) = 3x^2 - 1$, and so $f'(x_0) = f'(-1) = 2$. Furthermore, $f(x_0) = f(-1) = 1$. Then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{1}{2} = -\frac{3}{2}.$$

The tangent line should go through $(x_0, f(x_0)) = (-1, 1)$ and $(x_1, 0) = (-\frac{3}{2}, 0)$.

 Points earned: ______ out of 20.

10. Compute the following. Justify.

(a) (5 points) $\lim_{x\to 0} \frac{x^2+x}{e^x-1}$

Solution: We compute

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2+x}{e^x-1} &= \lim_{x\to 0} \frac{2x+1}{e^x} && \frac{0}{0} \text{ so use L'Hôpital} \\
&= 1 && \text{plug in } x = 0.
\end{aligned}$$

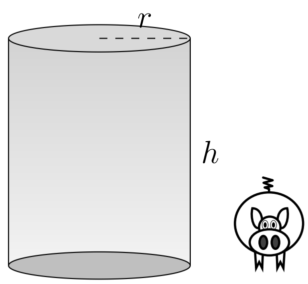
(b) (5 points) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Solution: We compute

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} && \text{combine fractions} \\
&= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} && \frac{0}{0} \text{ so use L'Hôpital} \\
&= \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} && \frac{0}{0} \text{ so use L'Hôpital} \\
&= \frac{-\sin 0}{\cos 0 + \cos 0 - 0\sin 0} && \text{plug in } x = 0 \\
&= 0.
\end{aligned}$$

 Points earned: ______ out of 10.

11. (10 points) Help Homer design a cylindrical silo to hold the output from his pet pig *Harry Plopper*. The silo should be able to hold 125π ft^3 of pig manure. Find the radius and height that will minimize the surface area of a sealed silo (top, bottom, and side).



Solution: We have that the volume V is $V = \pi r^2 h = 125\pi$. Therefore, $h = 125r^{-2}$. We want to minimize the area A given by

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r(125r^{-2}) = 2\pi r^2 + 250\pi r^{-1}.$$

Note that the domain of A is $0 < r < \infty$. We compute

$$\frac{dA}{dr} = 4\pi r - 250\pi r^{-2} = \frac{4\pi r^3 - 250\pi}{r^2},$$

which is never undefined, and it is equal to 0 on when $4\pi r^3 - 250\pi = 0$. Solving, we get the critical point $r = \sqrt[3]{\frac{125}{2}} = \frac{5}{\sqrt[3]{2}}$. Making a sign chart, we see that A is increasing to the right of $\frac{5}{\sqrt[3]{2}}$ and decreasing to the left of $\frac{5}{\sqrt[3]{2}}$. Thus the area is minimized when $r = \frac{5}{\sqrt[3]{2}}$. Since $h = 125r^{-2}$, we compute $h = 125(\frac{5}{\sqrt[3]{2}})^{-2} = 5\sqrt[3]{4}$. Thus we minimize the surface area when $r = \frac{5}{\sqrt[3]{2}}$ ft and $h = 5\sqrt[3]{4}$ ft.

 Points earned: ______ out of 10.