MAT 191 Test 3 October 28, 2013 (9:00–9:50 AM)

Name: ______________________ Academic Integrity Signature: ______________________

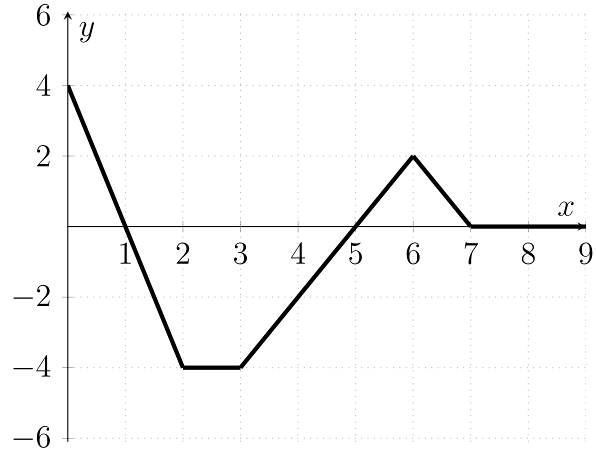
*I have abided by the UNCG Academic Integrity Policy.*

**Read all of the following information before starting the exam:**

- It is to your advantage to answer ALL of the 15 questions.
- It is your responsibility to make sure that you have all of the problems.
- There is no need to complete the test in order. The problems are independent.
- Correct numerical answers with insufficient justification may receive little or no credit.
- Clearly distinguish your final answer from your scratch work with a box or circle.
- *Budget your time!*
- If you have read all of these instructions, draw a happy face here.

| Page: | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 15 | 12 | 12 | 12 | 12 | 12 | 16 | 9 | 100 |
| Score: | | | | | | | | | |

1. The accompanying figure shows the velocity $y = v(t)$ in ft/sec of a particle moving on a line at time $t$ seconds for $0 \le t \le 9$.




(a) (3 points) When is the particle moving backwards?

> **Solution:** The particle is moving backwards when the velocity is negative. By looking at the graph, we see that this occurs between $t = 1$ second and $t = 5$ seconds. In other words, the particle is moving backwards on the interval $(1, 5)$ seconds.

(b) (3 points) When is the particle speeding up?

> **Solution:** When the velocity is positive, the particle is speeding up when the acceleration (rate of change of velocity) is positive. When the velocity is negative, the particle is speeding up when the acceleration is negative. Thus we see that the particle is speeding up on $(1, 2) \cup (5, 6)$ seconds.

(c) (3 points) When is the particle's acceleration positive?

> **Solution:** The acceleration is positive when the graph of velocity has positive slope. We see that this happens on the interval $(3, 6)$ seconds.

2. (6 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable, invertible function. Let $g(x) = f^{-1}(x)$ denote the inverse of $f$. Fill in the table below with the correct values. Write **N** if not enough information is given to compute the value.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 1 | 2 | 1 | $\frac{1}{3}$ |
| 1 | 0 | 3 | 0 | $\frac{1}{2}$ |
| 2 | 3 | 4 | $N$ | $N$ |
| 3 | $\frac{1}{2}$ | $N$ | 2 | $\frac{1}{4}$ |

**Solution:** Note that since $f$ an $g$ are inverse to each other, if $f(a) = b$, then $g(b) = a$ and $g'(b) = \frac{1}{f'(a)}$. USe this to fill out the table.

3. (6 points) A rock thrown vertically upward from the surface of the moon at a velocity 24 m/sec reaches a height of $s = 24t - 0.8t^2$ meters in $t$ seconds. How long does it take for the rock to reach its highest point?

**Solution:** The rock reaches its highest point when the velocity is zero

$$v = \frac{ds}{dt} = 24 - 1.6t = 0.$$

Solving for $t$, we have $t = 24/1.6 = 15$ sec.

4. (6 points) Find a formula for $\dfrac{dy}{dx}$ for the curve $x^3 + y^3 - 9xy = 0$.

**Solution:** Use implicit differentiation. We compute

$$\begin{aligned}
x^3 + y^3 - 9xy &= 0 \\
\frac{d}{dx}\left(x^3 + y^3 - 9xy\right) &= \frac{d}{dx}(0) \\
3x^2 + 3y^2\frac{dy}{dx} - 9(y + x\frac{dy}{dx}) &= 0 \\
3x^2 + 3y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} &= 0 \\
3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2 \\
(3y^2 - 9x)\frac{dy}{dx} &= 9y - 3x^2 \\
\frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x}
\end{aligned}$$

5. (6 points) Let $f(x) = x\ln(x)$. Compute $f'(e)$. Simplify your answer.

**Solution:** We compute using the product rule

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1.$$

Then

$$f'(e) = \ln(e) + 1 = 1 + 1 = 2.$$

6. (6 points) Use logarithmic differentiation to find $\frac{dy}{dx}$ if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1.$$

**Solution:** We compute

$$\begin{aligned}
y &= \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
\ln(y) &= \ln\left(\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right) \\
\ln(y) &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1) \\
\frac{d}{dx}(\ln(y)) &= \frac{d}{dx}\left(\ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1} \\
\frac{dy}{dx} &= y\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right) \\
\frac{dy}{dx} &= \left(\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right)\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right)
\end{aligned}$$

7. Compute the following.

   (a) (3 points) $\sec(\cos^{-1}(\frac{1}{2}))$

   **Solution:** We compute $\sec(\cos^{-1}(\frac{1}{2})) = \sec(\frac{\pi}{6}) = \frac{1}{\cos(\frac{\pi}{6})} = 2$.

   (b) (3 points) $\sin^{-1}(\sin(\frac{5\pi}{6}))$

   **Solution:** We compute $\sin^{-1}(\sin(\frac{5\pi}{6})) = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

8. (6 points) Compute $\dfrac{dy}{dt}$, where $y = \sin^{-1}(1-t)$.

**Solution:** We compute using the chain rule

$$\frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}.$$

9. (6 points) Complete the statement of the *First Derivative Test for Local Extrema*. Suppose $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$, except possibly at $c$ itself. Moving from left to right,

   1. if $f'$ changes from negative to positive at $c$, then $f$ has

      | local minimum | at $c$. |
      |---|---|

   2. if $f'$ changes from positive to negative at $c$, then $f$ has

      | local maximum | at $c$. |
      |---|---|

   3. if $f'$ does not change sign at $c$, then $f$ has

      | no local extremum | at $c$. |
      |---|---|

10. (6 points) Find the absolute maximum and absolute minimum of $g(x) = xe^x$ on the interval $-2 \leq x \leq 2$. Justify your answer. Make sure you also specify where the absolute maximum and absolute minimum occur.

**Solution:** First we find the critical points. We compute

$$g'(x) = e^x + xe^x = e^x(1 + x).$$

We see that $g'(x)$ is never undefined and $g'(x) = 0$ for $x = -1$. Thus we compute

$$\begin{aligned} g(-2) &= -2e^{-2} \\ g(2) &= 2e^2 \\ g(-1) &= -e^{-1}. \end{aligned}$$

Therefore the absolute maximum is $2e^2$ which occurs at $x = 2$. The absolute minimum is $-e^{-1}$ which occurs at $-1$. Note: Answering this way requires you to know (approximately) the value of $e$ so that you can recognize that $-e^{-1} < -2e^{-2}$. If you did not know this, you can get the same result by using the First Derivative Test. Namely, by making a sign chart, you can see that $g'(x) < 0$ for $x < 1$ and $g'(x) > 0$ for $x > 1$.

11. (a) (3 points) Clearly state the hypotheses of the Mean Value Theorem.

> **Solution:** If $f$ is a function that is continuous on the closed interval $[a, b]$ and differentiable on the interior $(a, b)$, then ...

(b) (3 points) For what values of $a$, $m$, and $b$ does the function

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ -x^2 + 3x + a & \text{if } 0 < x < 1 \\ mx + b & \text{if } 1 \leq x \leq 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

> **Solution:** The Mean Value Theorem would require $f$ to be continuous on $[0, 2]$ and differentiable on $(0, 2)$. We have $f(0) = 3$, so we need $\lim_{x \to 0^+} f(x) = a = 3$. Similarly, we have $f(1) = m+b$, so we need $\lim_{x \to 1^-} f(x) = -1+3+a = 5 = m+b$. We also need the tangent lines to match up at $x = 1$. In particular, we need $m$ to be equal to $-2x + 3$ evaluated at $x = 1$. In other words, we have $m = 1$. Putting it all together, we have
>
> $$a = 3, \quad m = 1, \quad \text{and} \quad b = 4.$$

12. Let $f(x) = x^2 + 4x + 1$.

(a) (3 points) What is the average rate of change of $f$ on $[0, 2]$?

> **Solution:** We compute $f(2) = 13$ and $f(0) = 1$. Thus
>
> $$\frac{\Delta y}{\Delta x} = \frac{13 - 1}{2 - 0} = 6.$$

(b) (3 points) Find $c$ in $(0, 2)$ so that $f'(c)$ equals the average rate of change you found in part (b).

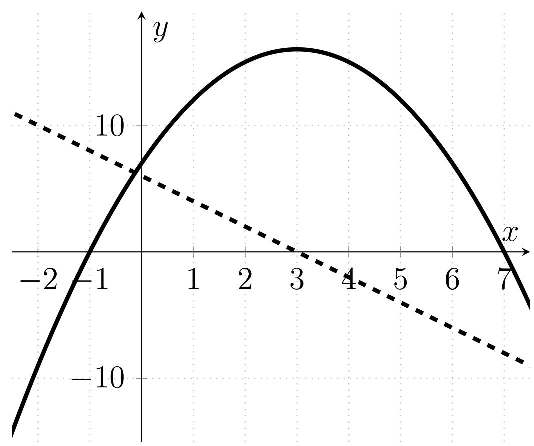
> **Solution:** We compute $f'(x) = 2x + 4$. Thus if $2c + 4 = 6$, we have $c = 1$.

 Points earned: ______ out of 12.

13. (10 points) Answer each question by circling True if it must be true and False if it is ever false. No justification is required.

- $\boxed{\text{True}}$ | False: Let $f$ be a differentiable function on $\mathbb{R}$. If the graph of $f$ is concave up on $\mathbb{R}$, then $f'$ is an increasing function.
- $\boxed{\text{True}}$ | False: If $f$ is a differentiable function which has a local maximum at an interior point $c$ of its domain, then $f'(c) = 0$.
- True | $\boxed{\text{False}}$: If $f$ is a continuous function on $\mathbb{R}$, then $f$ attains both an absolute maximum and absolute minimum value.
- $\boxed{\text{True}}$ | False: Suppose $f''$ is continuous on an open interval containing $c$. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
- True | $\boxed{\text{False}}$: Let $f$ be an decreasing function. Then $f''$ is negative.

14. The graphs below show the first (solid) and second (dashed) derivative of a function $y = f(x)$.




(a) (3 points) Where is the graph of $f$ both increasing and concave down?

**Solution:** The graph of $f$ is increasing where $f'$ is positive. It is concave down where $f''$ is negative. From the graphs that is the interval $(3, 7)$.

(b) (3 points) Identify where the local extrema of $f$ occur. For each, clearly identify whether it corresponds to a local maximum or local minimum.

**Solution:** The local extrema can occur at the critical points. We see that $f'(x) = 0$ at $x = -1$ and $x = 7$. We see that $f''(-1) > 0$ and $f''(7) < 0$. By the Second Derivative Test for Local Extrema, $f$ has a local minimum at $x = -1$ and a local maximum at $x = 7$.

15. Let $f(x) = x^3 - 3x + 3$.

(a) (2 points) Find the critical points of $f$.

**Solution:** We compute

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).$$

This is never undefined and equal to zero when $x = \pm 1$. The critical points are $\pm 1$.

(b) (2 points) Find the intervals where $f$ is increasing.

**Solution:** From (a), we have $f'(x) = 3(x+1)(x-1)$. Making a sign chart with dividing lines at $x = \pm 1$, we see that $f'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$. It follows that $f$ is increasing on $(-\infty, -1) \cup (1, \infty)$.

(c) (2 points) Find all the inflection points $(c, f(c))$ of $f$. Find the intervals where the graph $y = f(x)$ is concave up and those where it is concave down.

**Solution:** We compute $f''(x) = 6x$, which is zero at $x = 0$ and never undefined. Making a sign chart, we have that $f''(x) > 0$ for $x > 0$ and $f''(x) < 0$ for $x < 0$. It follows that the graph $y = f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

The graph changes concavity at $x = 0$. We compute $f(0) = 3$. It follows that the only inflection point is $(0, 3)$.

(d) (3 points) Identify all of the local extrema and where they occur. Clearly mark each as a local maximum or local minimum.

**Solution:** From (a), we have that the only critical points are where $x = 1$ and where $x = -1$. The second derivative is $f''(x) = 6x$. We compute $f''(-1) = -6 < 0$ and $f(-1) = 5$. By the Second Derivative Test for Local Extrema, we have a local maximum of 5 at $x = -1$. We compute $f''(1) = 6 > 0$ and $f(1) = 1$. By the Second Derivative Test for Local Extrema, we have a local minimum of 1 at $x = 1$.