

NON-INTEGRALITY OF SOME STEINBERG MODULES

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ABSTRACT. We prove that the Steinberg module of the special linear group of a quadratic imaginary number ring which is not Euclidean is not generated by integral apartments. Assuming the generalized Riemann hypothesis, this shows that the Steinberg module of a number ring is generated by integral apartments if and only if the ring is Euclidean. We also construct new cohomology classes in the top dimensional cohomology group of the special linear group of some quadratic imaginary number rings.

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1. INTRODUCTION

The cohomology of arithmetic groups has many applications to number theory and algebraic K -theory. One of the most useful tools for studying the high dimensional cohomology of $\mathrm{SL}_n(\mathcal{O}_K)$ for \mathcal{O}_K the ring of integers in a number field K is the Steinberg module $\mathrm{St}_n(K)$ which is a representation of $\mathrm{GL}_n(K)$. Let r_1 denote the number of real embeddings of K and r_2 denote the number of pairs of complex embeddings. Borel and Serre [BS73] proved that

$$H^{\nu_n - i}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) \cong H_i(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q} \otimes \mathrm{St}_n(K))$$

with

$$\nu_n = r_1 \frac{(n+1)n-2}{2} + r_2(n^2-1) - n + 1.$$

The number ν_n is the virtual cohomological dimension of $\mathrm{SL}_n(\mathcal{O}_K)$ and all rational cohomology groups (even with twisted coefficients) vanish above this degree.

To understand the cohomology in degree ν_n , it is important to understand generators for $\mathrm{St}_n(K)$ as an $\mathrm{SL}_n(\mathcal{O}_K)$ -module. There is a natural subset of $\mathrm{St}_n(K)$ called integral apartment classes and it is useful to know if these classes generated the entire Steinberg module. Let $\mathcal{T}_n(K)$ denote the Tits building of K , that is the geometric realization of the poset of proper nonempty subspaces of K^n ordered by inclusion. The Steinberg module is defined via:

$$\mathrm{St}_n(K) := \widetilde{H}_{n-2}(\mathcal{T}_n(K)).$$

Let \vec{L} denote a decomposition $K^n = L_1 \oplus \dots \oplus L_n$ into lines. Let $A_{\vec{L}}$ be the full subcomplex of $\mathcal{T}_n(K)$ with vertices direct sums of proper nonempty subsets of $\{L_1, \dots, L_n\}$. Each $A_{\vec{L}}$ is homeomorphic to an $(n-2)$ -sphere called an *apartment* and the image of the fundamental class of $A_{\vec{L}}$ in $\mathrm{St}_n(K) = \widetilde{H}_{n-2}(\mathcal{T}_n(K))$ is

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called an *apartment class*. We say that an apartment or apartment class is *integral* if

$$\mathcal{O}_K^n = (\mathcal{O}_K^n \cap L_1) \oplus \dots \oplus (\mathcal{O}_K^n \cap L_n).$$

In other words, \vec{L} is integral if it comes from a direct sum decomposition of \mathcal{O}_K^n into rank one projective submodules.

One of the main topics of this paper is the question:

For what number fields is $\text{St}_n(K)$ generated by integral apartment classes?

Integral apartments vanish after taking coinvariants by $\text{SL}_n(\mathcal{O}_K)$ for n larger than the class number so if $\text{St}_n(K)$ is generated by integral apartment classes, then $H^{\nu_n}(\text{SL}_n(\mathcal{O}_K); \mathbb{Q}) \cong H_0(\text{SL}_n(\mathcal{O}_K); \mathbb{Q} \otimes \text{St}_n(K))$ vanishes for n sufficiently large. In [AR79], Ash–Rudolph proved that $\text{St}_n(K)$ is generated by integral apartment classes if \mathcal{O}_K is Euclidean. Church–Farb–Putman proved that $H^{\nu_n}(\text{SL}_n(\mathcal{O}_K); \mathbb{Q})$ does not vanish if the class number of \mathcal{O}_K is greater than 1 [CFP15, Theorem D] and moreover that $\text{St}_n(K)$ is not generated by integral apartment classes for $n \geq 2$ for such rings [CFP15, Theorem B]. We prove the following.

Theorem 1.1. *Let \mathcal{O}_K be a quadratic imaginary number ring that is a PID but is not Euclidean. Then $\text{St}_n(K)$ is not generated by integral apartment classes.*

Let \mathcal{O}_d denote the ring of integers in $\mathbb{Q}(\sqrt{d})$. The only examples of rings satisfying the hypotheses of the above theorem are \mathcal{O}_d for $d = -19, -43, -67$ and -163 . However, assuming the generalized Riemann hypothesis, every number ring either has class number greater than 1, is Euclidean, or is quadratic imaginary [Wei73]. Thus we have the following corollary.

Corollary 1.2. *Let \mathcal{O}_K be a ring of integers in a number field K and consider $n \geq 2$. The generalized Riemann hypothesis implies that $\text{St}_n(K)$ is generated by integral apartments if and only if \mathcal{O}_K is Euclidean.*

For K quadratic imaginary, we have that $\nu_n = n^2 - n$. Our proof of Theorem 1.1 also gives the following.

Theorem 1.3. *For all n , we have:*

$$\dim_{\mathbb{Q}} H^{\nu_{2n}}(\text{SL}_{2n}(\mathcal{O}_d); \mathbb{Q}) \geq \begin{cases} 1 & \text{for } d = -43 \\ 2^n & \text{for } d = -67 \\ 6^n & \text{for } d = -163. \end{cases}$$

This shows that the rational cohomological dimension and the virtual cohomological dimension agree for these groups. This is the first example of homology in the virtual cohomological dimension of $\text{SL}_n(\mathcal{O}_K)$ for large n not coming from the class group. In particular, this gives the first example of the failure of [CFP15, Theorem D] to be sharp for large n .

Our proof involves using cohomology classes in $H^{\nu_2}(\text{SL}_2(\mathcal{O}_d); \mathbb{Q})$ to construct classes in $H^{\nu_{2n}}(\text{SL}_{2n}(\mathcal{O}_d); \mathbb{Q})$. In particular, the inequalities in Theorem 1.3 are actually equalities for $2n = 2$ by the work of Rahm [Rah13, Proposition 1]. This strategy does not apply to $H^{\nu_{2n}}(\text{SL}_{2n}(\mathcal{O}_{-19}); \mathbb{Q})$ since $H^{\nu_2}(\text{SL}_2(\mathcal{O}_{-19}); \mathbb{Q})$ vanishes. This also highlights the fact that our proof of nonintegrality does not rely on homological nonvanishing.

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2. POSETS

In this section, we begin by fixing notation for posets. Then we recall some connectivity results for complexes of unimodular vectors and their variants.

2.1. Notation.

Definition 2.1. Given a simplicial complex \mathcal{X} , there is an associated poset \mathbf{X} whose elements are the simplices of \mathcal{X} , ordered by inclusion.

In this paper we adopt the convention that we use calligraphic fonts for simplicial complexes and boldface fonts for their associated posets.

Definition 2.2. Given a poset \mathbf{Y} , let $\Delta(\mathbf{Y})$ denote the simplicial complex of non-degenerate simplices of the nerve of the poset.

Concretely, the p -simplices of $\Delta(\mathbf{Y})$ are given by ordered $(p+1)$ -tuples

$$\{y_0 < y_1 < \dots < y_p \mid y_i \in \mathbf{Y}\}$$

Define the *dimension* $\dim(\mathbf{Y})$ of \mathbf{Y} to be the dimension of $\Delta(\mathbf{Y})$. Let $|\mathbf{Y}|$ denote the geometric realization of $\Delta(\mathbf{Y})$. We refer to the *connectivity* of a poset or simplicial complex to mean the connectivity of its geometric realization.

We remark that, given a simplicial complex \mathcal{X} with associated poset \mathbf{X} , $\Delta(\mathbf{X})$ is the barycentric subdivision of \mathcal{X} . Thus they are not isomorphic in general but have homeomorphic geometric realizations.

Definition 2.3. Given a poset \mathbf{Y} and $y \in \mathbf{Y}$, let

$$\mathbf{Y}_{\leq y} := \{y' \in \mathbf{Y} \mid y' \leq y\}$$

and

$$\mathbf{Y}_{> y} := \{y' \in \mathbf{Y} \mid y' > y\}.$$

Define the *height* of y to be

$$\text{ht}(y) := \dim(\mathbf{Y}_{\leq y}).$$

This is one less than the length of a maximal length chain with supremum y .

2.2. The complex of partial frames.

In this subsection, we describe the complex of partial bases and a variant called the complex of partial frames. Here and in the rest of the paper, the symbol R will denote a commutative ring.

Definition 2.4. For a finite-rank free R -module V , we associate a simplicial complex $\mathcal{PB}(V)$ called the *complex of partial bases* of V . The vertices of $\mathcal{PB}(V)$ are primitive vectors in V , and vertices v_0, v_1, \dots, v_p span a p -simplex if and only if the vectors v_0, v_1, \dots, v_p are a *partial basis* for V , that is, a subset of a basis. When $V = R^n$, we will abbreviate this by \mathcal{PB}_n or by $\mathcal{PB}_n(R)$ when we want to emphasize the ring. We write $\mathbf{PB}(V)$, \mathbf{PB}_n , $\mathbf{PB}_n(R)$, to denote the posets associated to these simplicial sets; these are the posets of partial bases under inclusion.

Note that the complex $\mathcal{PB}(V)$ has dimension $(\text{rank}(V) - 1)$.

Definition 2.5. For V a finite rank free R -module, we write $\mathcal{B}(V)$ (similarly \mathcal{B}_n , $\mathcal{B}_n(R)$) for the simplicial complex defined as the quotient of $\mathcal{PB}(V)$ by identifying vertices v, u if the vectors $v, u \in V$ differ by multiplication by a unit. A p -simplex of $\mathcal{B}(V)$ encodes a decomposition of a direct summand of V into a direct sum of $(p+1)$ rank one free submodules of V . Following [CP17], we call $\mathcal{B}(V)$ the *complex of partial frames* of V . We write $\mathbf{B}(V)$, \mathbf{B}_n , or $\mathbf{B}_n(R)$, respectively, for the associated posets.

Definition 2.6. For V a finite rank free R -module, we write $\mathcal{B}'(V)$ (similarly \mathcal{B}'_n , $\mathcal{B}'_n(R)$) for the $(\text{rank}(V) - 2)$ -skeleton of $\mathcal{B}(V)$, and $\mathbf{B}'(V)$, \mathbf{B}'_n , or $\mathbf{B}'_n(R)$, respectively, for the associated posets.

Simplices of $\mathbf{B}'(V)$ consist of frames whose sum is not all of V .

Definition 2.7. Let R be an integral domain. For a finite-rank free R -module V , we write $\mathbf{T}(V)$ (similarly \mathbf{T}_n or $\mathbf{T}_n(R)$) for the poset of proper nonzero direct summands of V ordered by inclusion. The associated simplicial complex is the *Tits building* for V . We abbreviate $\Delta(\mathbf{T}(V))$, $\Delta(\mathbf{T}_n)$, and $\Delta(\mathbf{T}_n(R))$ by $\mathcal{T}(V)$, \mathcal{T}_n , and $\mathcal{T}_n(R)$ respectively.

Remark 2.8. Let V be a finite-rank free R -module and R an integral domain. Since R is an integral domain, it embeds in its field of fractions $\text{Frac}(R)$, and there is a natural bijection between the direct summands of V and the subspaces of the $\text{Frac}(R)$ -vector space $\text{Frac}(R) \otimes_R V$. In particular, there is a natural isomorphism $\mathbf{T}(V) \cong \mathbf{T}(\text{Frac}(R) \otimes_R V)$.

2.3. Connectivity results.

The following result is due to Solomon [Sol69] in the case K is finite; see also Quillen [Qui73, Theorem 2].

Proposition 2.9 (Solomon–Tits Theorem [Sol69]). *Let K be a field. Then $|\mathcal{T}_n(K)|$ has the homotopy type of a wedge of $(n - 2)$ -spheres.*

By Remark 2.8, $\mathcal{T}_n(R)$ is isomorphic to $\mathcal{T}_n(\text{Frac}(R))$ when R is an integral domain and hence it is also $(n - 3)$ -connected.

The following is straightforward and is the reason we primarily use \mathcal{B}_n instead of \mathcal{PB}_n .

Proposition 2.10. *Let R be a PID. Then $\mathcal{B}_1(R)$ is contractible.*

Proposition 2.11. *The graph $\mathcal{B}_2(R)$ is connected if and only if $\text{GL}_2(R)$ is generated by matrices of the form:*

$$\left\{ \left[\begin{array}{cc} u & * \\ 0 & v \end{array} \right], \left[\begin{array}{cc} u & 0 \\ * & v \end{array} \right], \left[\begin{array}{cc} 0 & u \\ v & 0 \end{array} \right] \mid u, v \in R^\times, * \in R \right\}.$$

Church–Farb–Putman [CFP15, Theorem 2.1] proved the analogous result for $\mathcal{PB}_2(R)$. This result follows from a standard argument in geometric group theory, which we state briefly below; see Church–Farb–Putman [CFP15, Proof of Proposition 2.7] and Serre [Ser80, Chapter 4].

Proof of Proposition 2.11. Let $E_2(R)$ denote the subgroup of $\text{GL}_2(R)$ generated by the given matrices. Let e_1, e_2 denote the standard basis for R^2 , and $e = (e_1, e_2)$ the associated edge in $\mathcal{B}_2(R)$. Notice that the generators of $E_2(R)$ are precisely those elements $g \in \text{GL}_2(R)$ such that $(g \cdot e) \cap e \neq \emptyset$. It follows that $E_2(R) \cdot e$ and $(\text{GL}_2(R) - E_2(R)) \cdot e$ are disjoint graphs. Moreover, since $\text{GL}_2(R)$ acts transitively on the edges of $\mathcal{B}_2(R)$, the graph $\mathcal{B}_2(R)$ is the union of $E_2(R) \cdot e$ and $(\text{GL}_2(R) - E_2(R)) \cdot e$, and $E_2(R) \cdot e$ must be the connected component containing the edge e . Hence $E_2(R)$ is precisely the stabilizer of this connected component, and $\mathcal{B}_2(R)$ is connected if and only if $\text{GL}_2(R) = E_2(R)$. \square

Combining Proposition 2.11 with Cohn [Coh66, Theorem 6.1 and 6.2] implies the following result.

Proposition 2.12. *Let \mathcal{O}_K be a quadratic imaginary number ring that is a PID but is not Euclidean. Then $\mathcal{B}_2(\mathcal{O}_K)$ is not connected.*

Proposition 2.13. *If $\mathcal{PB}_n(R)$ is d -connected for some ring R , so is $\mathcal{B}_n(R)$.*

Proof. Consider the natural projection $\mathcal{PB}_n(R) \rightarrow \mathcal{B}_n(R)$. We can construct a splitting $\mathcal{B}_n(R) \rightarrow \mathcal{PB}_n(R)$ by choosing a primitive vector v for each line in R^n . Hence $\pi_i(|\mathcal{PB}_n(R)|) \rightarrow \pi_i(|\mathcal{B}_n(R)|)$ is surjective for all i . \square

Combining this with a result of van der Kallen [vdK80, Theorem 2.6 (i)] gives the following corollary.

Corollary 2.14. *Let R be a PID. Then $\mathcal{B}_n(R)$ is $(n - 3)$ -connected.*

Proof. Since PIDs satisfy the Bass Stable Range condition SR_3 , the work of van der Kallen [vdK80, Theorem 2.6 (i)] implies $\mathcal{PB}_n(R)$ is $(n - 3)$ -connected. The claim now follows from Proposition 2.13. \square

Corollary 2.15. *Let R be a PID. $\mathcal{B}'_n(R)$ is $(n - 3)$ -connected.*

Proof. $\mathcal{B}'_n(R)$ is the $(n - 2)$ -skeleton of $\mathcal{B}_n(R)$, which is $(n - 3)$ -connected. The claim follows from simplicial approximation. \square

3. THE MAP OF POSETS SPECTRAL SEQUENCE

In this section we recall a useful spectral sequence arising from maps of posets.

3.1. Homology of posets.

We begin by defining the homology of a poset with coefficients in a functor F .

Definition 3.1. Given a poset \mathbf{Y} , and a functor F from the poset \mathbf{Y} (viewed as a category) to the category $\underline{\text{Ab}}$ of abelian groups, define the chain groups

$$C_p(\mathbf{Y}; F) := \bigoplus_{y_0 < \cdots < y_p \in \mathbf{Y}} F(y_0)$$

and differential given by the alternating sum of the face maps

$$\begin{aligned} d_i : \bigoplus_{y_0 < \cdots < y_p} F(y_0) &\longrightarrow \bigoplus_{y_0 < \cdots < \hat{y}_i < \cdots < y_p} F(y_0) & (i \neq 0) \\ d_0 : \bigoplus_{y_0 < \cdots < y_p} F(y_0) &\longrightarrow \bigoplus_{y_1 < \cdots < y_p} F(y_1). \end{aligned}$$

Here, the map d_i with $(i \neq 0)$ maps the summand indexed by $(y_0 < \cdots < y_p)$ to the summand indexed by $(y_0 < \cdots < \hat{y}_i < \cdots < y_p)$, and acts by the identity on the group $F(y_0)$. The map d_0 maps the summand indexed by $(y_0 < \cdots < y_p)$ to the summand indexed by $(y_1 < \cdots < y_p)$, and the map $F(y_0) \rightarrow F(y_1)$ is the image of the morphism $y_0 < y_1 \in \mathbf{Y}$ under the functor F .

If $F = \mathbb{Z}$ is the constant functor with identity maps, then $H_*(\mathbf{Y}; \mathbb{Z})$ coincides with the homology groups $H_*(|\mathbf{Y}|)$.

The following lemma is adapted from Charney [Cha87, Lemma 1.3].

Lemma 3.2. *Suppose that $F : \mathbf{Y} \rightarrow \underline{\text{Ab}}$ is a functor supported on elements of height m . Then*

$$H_p(\mathbf{Y}; F) = \bigoplus_{\text{ht}(y_0)=m} \tilde{H}_{p-1}(\mathbf{Y}_{>y_0}; F(y_0)).$$

Proof. Suppose that $F : \mathbf{Y} \rightarrow \underline{\text{Ab}}$ is supported on elements of height m .

$$\begin{aligned} C_p(\mathbf{Y}; F) &= \bigoplus_{y_0 < \cdots < y_p \in \mathbf{Y}} F(y_0) \\ &= \bigoplus_{\substack{y_0 < \cdots < y_p \in \mathbf{Y} \\ \text{ht}(y_0)=m}} F(y_0) \\ &\cong \bigoplus_{\text{ht}(y_0)=m} \left(F(y_0) \otimes_{\mathbb{Z}} \bigoplus_{y_0 < \cdots < y_p} \mathbb{Z} \right) \\ &\cong \bigoplus_{\text{ht}(y_0)=m} \left(F(y_0) \otimes_{\mathbb{Z}} \tilde{C}_{p-1}(\mathbf{Y}_{>y_0}; \mathbb{Z}) \right). \end{aligned}$$

Thus

$$H_p(\mathbf{Y}; F) = \bigoplus_{\text{ht}(y_0)=m} \tilde{H}_{p-1}(\mathbf{Y}_{>y_0}; F(y_0)). \quad \square$$

3.2. The spectral sequence for a map of posets.

Given a map of posets $f : \mathbf{X} \rightarrow \mathbf{Y}$, there is an associated spectral sequence introduced by Quillen [Qui78, Section 7]; see also Charney [Cha87, Section 1].

Definition 3.3. Let $f : \mathbf{X} \rightarrow \mathbf{Y}$ be a map of posets. For $y \in \mathbf{Y}$, define $f \setminus y \subseteq \mathbf{X}$ to be the subposet of elements whose images in \mathbf{Y} are less than or equal to y :

$$f \setminus y := \{x \in \mathbf{X} \mid f(x) \leq y\}.$$

Theorem 3.4. *Given a map of posets $f : \mathbf{X} \rightarrow \mathbf{Y}$, there is a spectral sequence*

$$E_{p,q}^2 = H_p(\mathbf{Y}; [y \mapsto H_q(f \setminus y)]) \implies H_{p+q}(\mathbf{X}).$$

This spectral sequence is an instance of the Grothendieck spectral sequence for the composition of functors:

$$\begin{aligned} \text{Fun}(\mathbf{X}, \underline{\mathbf{Ab}}) &\longrightarrow \text{Fun}(\mathbf{Y}, \underline{\mathbf{Ab}}) \\ F &\longmapsto [y \mapsto H_0(f/y; F)] \\ \text{Fun}(\mathbf{Y}, \underline{\mathbf{Ab}}) &\longrightarrow \underline{\mathbf{Ab}} \\ F' &\longmapsto H_0(\mathbf{Y}, F'). \end{aligned}$$

4. NON-INTEGRALITY

In this section, we use the map of poset spectral sequence and connectivity results to prove that the Steinberg modules of quadratic imaginary PIDs which are not Euclidean are not generated by integral apartment classes. We begin with the following lemma on poset homology.

Lemma 4.1. *Let R be a PID. Then $H_p(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) = 0$ unless $p = n - 3$, when*

$$H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) \cong \bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}_{n-2} \otimes \tilde{H}_0(\mathbf{B}_2).$$

In particular, if \mathbf{B}_2 is not connected and $n \geq 3$, the group $H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-)))$ is not zero.

Proof. The functor $\tilde{H}_0(\mathbf{B}(-)): \mathbf{T}_n \rightarrow \mathbf{Ab}$ is nonzero except possibly on submodules $V \subseteq R^n$ of rank 2 by [Proposition 2.10](#) and [Corollary 2.14](#). Then by [Lemma 3.2](#), we find

$$H_p(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) = \bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \tilde{H}_{p-1}(\mathbf{T}(R^n/V); \tilde{H}_0(\mathbf{B}(V))).$$

Then $\mathbf{T}(R^n/V)$ is spherical of dimension $(n-2)-2$ by the Solomon–Tits theorem and we find

$$H_p(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) = \begin{cases} \bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}(R^n/V) \otimes \tilde{H}_0(\mathbf{B}(V)), & p = n - 3 \\ 0, & \text{otherwise} \end{cases}$$

as claimed. \square

Consider the map of posets

$$\begin{aligned} f: \mathbf{B}'_n &\longrightarrow \mathbf{T}_n \\ \{v_0, \dots, v_p\} &\longmapsto \text{span}_R(v_0, \dots, v_p). \end{aligned}$$

For the remainder of the section we let $E_{p,q}^r$ denote the spectral sequence associated to this map.

Observe that, for $V \in \mathbf{T}_n$, the subposet $f \setminus V$ is precisely $\mathbf{B}(V)$. Thus the spectral sequence associated to f satisfies

$$E_{p,q}^2 = H_p(\mathbf{T}_n; [V \mapsto H_q(\mathbf{B}(V))]) \implies H_{p+q}(\mathbf{B}'_n).$$

Proposition 4.2. *Let $n \geq 2$, R be a PID, and $E_{p,q}^r$ denote the spectral sequence associated to the map of posets $f: \mathbf{B}'_n \rightarrow \mathbf{T}_n$. Then*

- (i) $E_{p,q}^\infty = 0$ for $(p+q) \neq (n-2)$,
- (ii) $E_{p,q}^2 = 0$ unless $(p+q) = (n-2)$, $(p+q) = (n-3)$, or $(p,q) = (0,0)$,
- (iii) $E_{n-3,0}^2 = 0$.

Proof. Since the spectral sequence converges to $H_{p+q}(\mathbf{B}'_n)$ and $\dim(\mathbf{B}'_n) = n-2$, part (i) follows from the fact that $\mathbf{B}'_n(R)$ is $(n-3)$ -connected by [Corollary 2.15](#). Part (iii) follows from parts (i) and (ii): part (ii) implies that for $r \geq 2$ there are no nontrivial differentials to or from the group $E_{n-3,0}^r$, and so by part (i) we conclude that $E_{n-3,0}^2 \cong 0$.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_p(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) & \longrightarrow & H_p(\mathbf{T}_n; H_0(\mathbf{B}(-))) & \longrightarrow & H_p(\mathbf{T}_n; \mathbb{Z}) \longrightarrow \cdots \\
& & \parallel & & \parallel & & \parallel \\
& & \left\{ \begin{array}{l} \bigoplus_{\text{rank}(V)=2} \text{St}_{n-2} \otimes \tilde{H}_0(\mathbf{B}_2), \quad p = n-3 \\ 0, \quad \text{otherwise} \end{array} \right. & & E_{p,0}^2 & & \left\{ \begin{array}{l} \text{St}_n, \quad p = n-2 \\ \mathbb{Z}, \quad p = 0 \\ 0, \quad \text{otherwise} \end{array} \right.
\end{array}$$

Again we conclude that $E_{p,0}^2$ vanishes unless p is $(n-3)$, $(n-2)$, or 0 , which completes the proof of part (ii). \square

Proposition 4.3. *Let $n \geq 3$ and R be a PID. There is an exact sequence:*

$$0 \rightarrow E_{n-2,0}^2 \rightarrow \text{St}_n \rightarrow H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) \rightarrow 0.$$

Proof. Consider again the short exact sequence of functors

$$0 \rightarrow \tilde{H}_0(\mathbf{B}(-)) \rightarrow H_0(\mathbf{B}(-)) \rightarrow \mathbb{Z} \rightarrow 0$$

and the associated long exact sequence on the homology of \mathbf{T}_n described in the proof of [Proposition 4.2](#). When $p = (n-2)$, we get the following long exact sequence:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_{n-2}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) & \longrightarrow & E_{n-2,0}^2 & \longrightarrow & H_{n-2}(\mathbf{T}_n; \mathbb{Z}) & \longrightarrow & H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) & \longrightarrow & E_{n-3,0}^2 \longrightarrow \cdots \\
& & \parallel & & & & \parallel & & \parallel & & \parallel \\
& & 0 & & & & \text{St}_n & & \bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}_{n-2} \otimes \tilde{H}_0(\mathbf{B}_2) & & 0
\end{array}$$

Here, the vanishing of $E_{n-3,0}^2$ follows from [Proposition 4.2](#) part (iii), and the groups $H_{n-2}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-)))$ and $H_{n-2}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-)))$ are computed in [Lemma 4.1](#). We obtain the desired short exact sequence. \square

Since

$$H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) \cong \bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}_{n-2} \otimes \tilde{H}_0(\mathbf{B}_2)$$

by [Lemma 4.1](#), the short exact sequence of [Proposition 4.3](#) has the following consequence.

Corollary 4.4. *Let $n \geq 3$ and R be a PID with \mathbf{B}_2 is not connected. $E_{n-2,0}^2 \rightarrow \text{St}_n$ is not surjective.*

There is an edge morphism $H_{n-2}(\mathbf{B}'_n) \rightarrow E_{n-2,0}^\infty$. Because there are no differentials into $E_{n-2,0}^r$ for $r > 1$, there is a map $E_{n-2,0}^\infty \rightarrow E_{n-2,0}^2$. The following proposition is implicit in the proof of Church–Farb–Putman [[CFP15](#), Proof of Theorem A]. In particular, see Equation 3.1 and the surrounding discussion.

Proposition 4.5. *The composition $H_{n-1}(\mathbf{B}_n, \mathbf{B}'_n) \rightarrow H_{n-2}(\mathbf{B}'_n) \rightarrow E_{n-2,0}^\infty \rightarrow E_{n-2,0}^2 \rightarrow \text{St}_n$ is the map sending a frame to the associated integral apartment class.*

Proposition 4.6. *Take $n \geq 3$ and let R be PID with \mathbf{B}_2 not connected. The composition $H_{n-1}(\mathbf{B}_n, \mathbf{B}'_n) \rightarrow H_{n-2}(\mathbf{B}'_n) \rightarrow E_{n-2,0}^\infty \rightarrow E_{n-2,0}^2 \rightarrow \text{St}_n$ is not surjective.*

Proof. The map $E_{n-2,0}^2 \rightarrow \text{St}_n$ is not surjective so the composition is not surjective. \square

Proposition 4.7. *Let R be a PID with \mathbf{B}_2 not connected. The map $H_1(\mathbf{B}_2, \mathbf{B}'_2) \rightarrow \text{St}_2$ is not surjective.*

Proof. Since R is PID, $\mathbf{B}'_2 \cong \mathbf{T}_2$ so we just need to show the map $H_1(\mathbf{B}_2, \mathbf{B}'_2) \rightarrow \tilde{H}_0(\mathbf{B}'_2)$ is not surjective. This fits into a long exact sequence:

$$H_1(\mathbf{B}_2, \mathbf{B}'_2) \rightarrow \tilde{H}_0(\mathbf{B}'_2) \rightarrow \tilde{H}_0(\mathbf{B}_2).$$

Since $\tilde{H}_0(\mathbf{B}_2)$ is not zero, $H_1(\mathbf{B}_2, \mathbf{B}'_2) \rightarrow \tilde{H}_0(\mathbf{B}'_2)$ is not surjective. \square

We now prove [Theorem 1.1](#).

Proof of Theorem 1.1. Let \mathcal{O}_K be a quadratic imaginary number ring which is a PID but not Euclidean. By [Proposition 2.12](#), \mathbf{B}_2 is not connected. By [Proposition 4.6](#) for $n \geq 3$ and [Proposition 4.7](#) for $n = 2$, the map

$$H_{n-1}(\mathbf{B}_n, \mathbf{B}'_n) \rightarrow \mathbf{St}_n$$

is not surjective. By [Proposition 4.5](#) in the case $n \geq 3$ and inspection in the case $n = 2$, its image is the submodule of the Steinberg module generated by integral apartments. Thus, $\mathbf{St}_n(K)$ is not generated by integral apartment classes. \square

Remark 4.8. The arguments show the Steinberg module of any PID is not generated by integral apartment classes if \mathbf{B}_2 not connected. See and Cohn [[Coh66](#), Theorem C] and Church–Farb–Putman [[CFP15](#), Proof of Proposition 2.1] for examples of rings of integers in function fields with \mathbf{B}_2 not connected.

Remark 4.9. In this section, we used the map of posets spectral sequence to prove a certain map is not surjective using that a particular simplicial complex is not connected. This is an adaptation of the arguments of [[MPP](#)] where a conjecture of Lee–Szczarba [[LS76](#), Page 28] is disproved. There, the fact that a certain simplicial complex is not simply connected is used to show a map is not injective.

5. NONVANISHING OF TOP DEGREE COHOMOLOGY

In this section, we show that our proof of non-integrality can sometimes be adapted to show non-vanishing of the cohomology in the virtual cohomological dimension. Throughout, d will denote a negative squarefree integer.

5.1. An equivariant calculation of $H^2(\mathrm{SL}_2(\mathcal{O}_d); \mathbb{Q})$ for $d = -43, -67, -163$.

We begin by recalling a calculation of $H^2(\mathrm{SL}_2(\mathcal{O}_d); \mathbb{Q})$ for $d = -43, -67, -163$ and then describe our calculation of $H^2(\mathrm{GL}_2(\mathcal{O}_d); \mathbb{Q})$ for these rings. We also compute the torsion at primes greater than three. Note that $\nu_2 = 2$ so this is the virtual cohomological dimension. Since the units in these rings are $\{-1, 1\}$, $H^i(\mathrm{SL}_n(\mathcal{O}_d))$ is naturally a $\mathbb{Z}/2$ -representation. Knowing both $H^i(\mathrm{SL}_n(\mathcal{O}_d); \mathbb{Q})$ and $H^i(\mathrm{GL}_n(\mathcal{O}_d); \mathbb{Q})$ allows one to compute $H^i(\mathrm{SL}_n(\mathcal{O}_d); \mathbb{Q})$ as a $\mathbb{Z}/2$ -representation.

The following was proven by Rahm [[Rah13](#), Proposition 1], with some cases previously known by the work of Vogtmann [[Vog85](#)]. Rahm’s result concerns the integer homology of the group $\mathrm{PSL}_2(\mathcal{O}_d)$, whose rational homology agrees with that of $\mathrm{SL}_2(\mathcal{O}_d)$.

Theorem 5.1 (Rahm [[Rah13](#), Proposition 1]). *Let \mathcal{O}_d denote the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{d})$. Then*

$$\dim_{\mathbb{Q}} H^2(\mathrm{SL}_n(\mathcal{O}_d); \mathbb{Q}) \geq \begin{cases} 1 & \text{for } d = -43 \\ 2 & \text{for } d = -67 \\ 6 & \text{for } d = -163. \end{cases}$$

We now describe the analogous calculation for $\mathrm{GL}_2(\mathcal{O}_d)$. This will follow from the methods of [[DSGG⁺16](#)], [[EVGS13](#), §3], [[Sou00](#), §2], and [[AGM11](#)]. For any positive integer b , let \mathcal{S}_b be the Serre class of finite abelian groups with orders only divisible by primes less than or equal to b [[Ser53](#)]. Let Γ be a finite index subgroup in $\mathrm{GL}_n(\mathcal{O}_K)$. If b is larger than all the primes dividing the orders of finite subgroups of Γ , then modulo \mathcal{S}_p the group cohomology of Γ can be computed using the of the Voronoi complex, a chain complex generated by cells of positive definite Hermitian forms. In particular, up to

torsion divisible by primes less than or equal to b , the Voronoi complex captures the group cohomology. We refer the reader to [EVGS13, §3.1] and [DSGG⁺16, §3] for the precise definition.

Theorem 5.2 ([DSGG⁺16, Theorem 3.7]). *Let b be an upper bound on the torsion primes for $\mathrm{GL}_n(\mathcal{O}_d)$. Modulo the Serre class \mathcal{S}_b ,*

$$H_i(\mathrm{Vor}_{n,d}) \cong H_{i-(n-1)}(\mathrm{GL}_n(\mathcal{O}_d); \mathrm{St}_n(\mathbb{Q}(\sqrt{d}))) \cong H^{n^2-1-i}(\mathrm{GL}_n(\mathcal{O}_d)).$$

By [DSGG⁺16, Lemma 3.9], the torsion primes for $\mathrm{GL}_2(\mathcal{O}_d)$ are 2 and 3.

Corollary 5.3. *Modulo \mathcal{S}_3 ,*

$$H_1(\mathrm{Vor}_{2,d}) \cong H_0(\mathrm{GL}_2(\mathcal{O}_d); \mathrm{St}_2(\mathbb{Q}(\sqrt{d}))) \cong H^2(\mathrm{GL}_2(\mathcal{O}_d)).$$

Theorem 5.4.

$$H_1(\mathrm{Vor}_{2,-43}) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_1(\mathrm{Vor}_{2,-67}) \cong (\mathbb{Z}/2\mathbb{Z})^2, \quad H_1(\mathrm{Vor}_{2,-163}) \cong (\mathbb{Z}/2\mathbb{Z})^6.$$

Proof. The algorithms for computing Voronoi homology are given in [DSGG⁺16, §6]. The three stages are:

- (1) determining the perfect forms;
- (2) computing the Voronoi complex and the differentials;
- (3) computing the homology.

We implemented these steps using MAGMA [BCP97] and lrs Vertex Enumeration/Convex Hull package [Avi00], a C implementation of the reverse search algorithm for vertex enumeration/convex hull problems. We include some details to give a sense of the computational task. We remark the determination of perfect forms is already known [Yas10], and the current computational results are consistent with the earlier results.

Let $\mathcal{H}^2(\mathbb{C})$ denote the 4-dimensional real vector space of 2×2 Hermitian matrices with complex coefficients. Using the chosen complex embedding of $K = \mathbb{Q}(\sqrt{d})$ we can view $\mathcal{H}^2(K)$, the Hermitian matrices with coefficients in K , as a subset of $\mathcal{H}^2(\mathbb{C})$. Moreover, this embedding allows us to view $\mathcal{H}^2(\mathbb{C})$ as a \mathbb{Q} -vector space such that the rational points of $\mathcal{H}^2(\mathbb{C})$ are exactly $\mathcal{H}^2(K)$. Let $C^* \subset \mathcal{H}^2(\mathbb{C})$ denote the nonzero positive semi-definite Hermitian forms with K -rational kernel, and let X denote the quotient of C^* by positive homothety. There is a natural identification of a subset of X^* with hyperbolic 3-space \mathbb{H}^3 . Voronoi theory describes a decomposition of X^* in terms of configurations of minimal vectors of Hermitian forms, which gives rise to a tessellation of \mathbb{H}^3 by ideal 3-dimensional hyperbolic polytopes. These polytopes with certain gluing maps determine the Voronoi complex and differentials.

The computations for $d = -67$ and $d = -163$ are larger, so we just summarize some of the key features after giving details for the case $d = -43$.

Let $\omega = \frac{1+\sqrt{43}}{2}$. Consider the vectors v_1, v_2, \dots, v_{21} :

$$\begin{aligned} & \begin{bmatrix} -3\omega + 3 \\ 2\omega - 12 \end{bmatrix}, \begin{bmatrix} -\omega + 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ -\omega - 2 \end{bmatrix}, \begin{bmatrix} 2\omega + 7 \\ -4\omega + 2 \end{bmatrix}, \begin{bmatrix} -\omega + 10 \\ -2\omega - 10 \end{bmatrix}, \begin{bmatrix} \omega \\ -\omega + 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\ & \begin{bmatrix} \omega + 1 \\ -\omega + 2 \end{bmatrix}, \begin{bmatrix} -\omega + 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ -\omega - 3 \end{bmatrix}, \begin{bmatrix} \omega \\ -\omega + 2 \end{bmatrix}, \begin{bmatrix} \omega + 1 \\ -\omega + 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -\omega - 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -\omega - 2 \end{bmatrix}, \\ & \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\omega + 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -\omega + 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \omega + 2 \\ -\omega + 2 \end{bmatrix}, \begin{bmatrix} -\omega + 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ -\omega - 1 \end{bmatrix}. \end{aligned}$$

Using the Voronoi algorithm adapted to this case, we find that there are four equivalence classes of perfect forms. We describe a perfect form by its set of minimal vectors (up to ± 1) by giving the indices of the vectors that are the minimal vectors for that form. For example, $\{1, 2, 5\}$ represents a form with

minimal vectors $\{\pm v_1, \pm v_2, \pm v_5\}$. We find explicit representatives for each class of perfect forms:

$$\begin{aligned}\phi_1 &= \{1, 2, 3, 4, 5, 6\}, \\ \phi_2 &= \{6, 7, 8, 9, 10, 11\}, \\ \phi_3 &= \{2, 3, 6, 7, 8, 12, 13, 14, 15\}, \\ \phi_4 &= \{7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}.\end{aligned}$$

These perfect forms determine hyperbolic polytopes. The facets of ϕ_1 are

$$\{1, 3, 5, 6\}, \{2, 4, 5, 6\}, \{1, 2, 3, 4\}, \{2, 3, 6\}, \{1, 4, 5\}.$$

The facets of ϕ_2 are

$$\{3, 4, 5\}, \{2, 3, 4, 6\}, \{1, 4, 5, 6\}, \{1, 2, 3, 5\}, \{1, 2, 6\}.$$

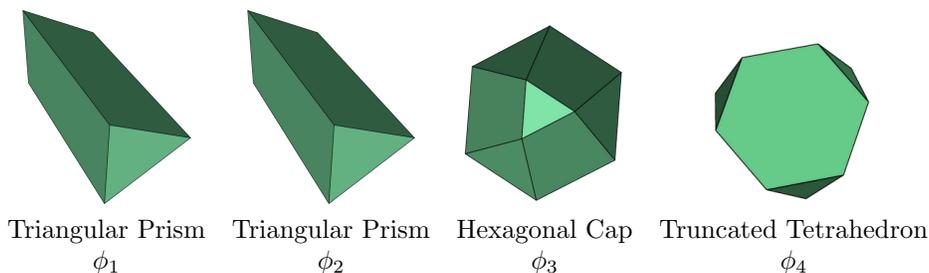
The facets of ϕ_3 are

$$\{1, 2, 3, 6, 7, 8\}, \{1, 2, 9\}, \{5, 6, 8\}, \{1, 5, 8, 9\}, \{2, 4, 7, 9\}, \{4, 5, 9\}, \{3, 4, 5, 6\}, \{3, 4, 7\}.$$

The facets of ϕ_4 are

$$\begin{aligned}\{1, 3, 4, 9, 10, 12\}, \{2, 5, 6, 9, 11, 12\}, \{4, 5, 7, 8, 10, 11\}, \{6, 7, 11\}, \\ \{1, 2, 3, 6, 7, 8\}, \{1, 2, 12\}, \{3, 8, 10\}, \{4, 5, 9\}.\end{aligned}$$

We see that ϕ_1 and ϕ_2 give rise to triangular prisms, ϕ_3 gives rise to a hexagonal cap, and ϕ_4 gives rise to a truncated tetrahedron.



The polytopes are given explicitly, and the gluing maps can be computed. There are four types of 3-dimensional cells, six types of 2-dimensional faces, and four types of 1-dimensional edges. We have $H_1(\text{Vor}_{2,-43})$ is the cokernel of the differential from 2-cells to 1-cells, and an explicit linear algebra computation gives the result.

For $d = -67$, there are seven equivalence classes of perfect forms that give rise to one octahedron, two triangular prisms, one hexagonal cap, two square pyramids, and one truncated tetrahedron. This gives seven types of 3-dimensional cells, thirteen types of 2-dimensional faces, and eight types of 1-dimensional edges. Again, we compute the differential from 2-cells to 1-cells, and an explicit linear algebra computation gives the result.

For $d = -163$, there are twenty-five equivalence classes of perfect forms that give rise to eleven tetrahedra, one cuboctahedron, eight triangular prisms, two hexagonal caps, and three square pyramids. This gives twenty-five types of 3-dimensional cells, forty-nine types of 2-dimensional faces, and twenty-seven types of 1-dimensional edges. Again, we compute the differential from 2-cells to 1-cells, and an explicit linear algebra computation gives the result. \square

Corollary 5.5. For $d \in \{-43, -67, -163\}$, modulo \mathcal{S}_3 ,

$$H_0(\text{GL}_2(\mathcal{O}_d); \text{St}_n(\mathbb{Q}(\sqrt{d}))) \cong H^2(\text{GL}_2(\mathcal{O}_d)) = 0.$$

5.2. Nonvanishing for $2n \geq 4$.

The goal of this subsection is to leverage the calculations of the previous section to prove [Theorem 1.3](#).

Lemma 5.6. *Let R be a PID. We have that $\tilde{H}_0(\mathcal{B}_2; \mathbb{Q})_{\mathrm{SL}_2(R)} \cong (\mathrm{St}_2 \otimes \mathbb{Q})_{\mathrm{SL}_2(R)}$.*

Proof. We will first show that $H_1(\mathcal{B}_2, \mathcal{B}'_2; \mathbb{Q})_{\mathrm{SL}_2(R)} \cong 0$ by the usual argument showing integrality implies homological vanishing. Note that $H_1(\mathcal{B}_2, \mathcal{B}'_2)$ is the free abelian group on frames (F_1, F_2) in R^2 . Let g be an element of $\mathrm{SL}_2(R)$ with $g(F_1) = F_2$ and $g(F_2) = F_1$. We have that g acts via multiplication by negative -1 on (F_1, F_2) (see also [\[CFP15, proof of Theorem C\]](#)). Thus $(F_1, F_2) = -(F_1, F_2)$ in $H_1(\mathcal{B}_2, \mathcal{B}'_2; \mathbb{Q})_{\mathrm{SL}_2(R)}$. Since 2 is invertible in \mathbb{Q} , this element and hence the group vanishes.

Since \mathcal{B}'_2 is the zero skeleton of \mathcal{B}_2 , $H_0(\mathcal{B}_2, \mathcal{B}'_2; \mathbb{Q}) \cong 0$. Since R is a PID, $\mathcal{B}'_2 \cong \mathcal{T}_2$. Thus, $\tilde{H}_0(\mathcal{B}'_2) \cong \mathrm{St}_2$. The claim now follows from applying the coinvariants functor to the exact sequence:

$$H_1(\mathcal{B}_2, \mathcal{B}'_2; \mathbb{Q}) \rightarrow \tilde{H}_0(\mathcal{B}'_2; \mathbb{Q}) \rightarrow \tilde{H}_0(\mathcal{B}_2; \mathbb{Q}) \rightarrow H_0(\mathcal{B}_2, \mathcal{B}'_2; \mathbb{Q}).$$

□

Combining results from the previous section gives the following.

Lemma 5.7. *Let R be a PID. There is a surjection*

$$\mathrm{St}_n(R)_{\mathrm{SL}_n(R)} \longrightarrow \left(\bigoplus_{\substack{V \subseteq R^n \\ \mathrm{rank}(V)=2}} \mathrm{St}(R^n/V) \otimes \tilde{H}_0(\mathbf{B}(V)) \right)_{\mathrm{SL}_n(R)}.$$

This surjection is equivariant with respect to the action of $\mathbb{Z}/2 \cong \begin{bmatrix} \pm 1 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{bmatrix} \subseteq \mathrm{GL}_n(R)$.

Proof. By [Lemma 4.1](#) and [Proposition 4.3](#), there is a $\mathrm{GL}_n(R)$ -equivariant surjection

$$\mathrm{St}_n(R) \longrightarrow H_{n-3}(\mathbf{T}_n; \tilde{H}_0(\mathbf{B}(-))) \cong \bigoplus_{\substack{V \subseteq R^n \\ \mathrm{rank}(V)=2}} \mathrm{St}(R^n/V) \otimes \tilde{H}_0(\mathbf{B}(V)).$$

The claim follows from the right-exactness of coinvariants. □

Lemma 5.8. *Let R be a PID with group of units $\{\pm 1\}$. There is an isomorphism*

$$\left(\bigoplus_{\substack{V \subseteq R^n \\ \mathrm{rank}(V)=2}} \mathrm{St}(R^n/V) \otimes \tilde{H}_0(\mathbf{B}(V)) \right)_{\mathrm{SL}_n(R)} \cong (\mathrm{St}_{n-2}(R))_{\mathrm{SL}_{n-2}(R)} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (\tilde{H}_0(\mathbf{B}_2))_{\mathrm{SL}_2(R)},$$

and this isomorphism is equivariant with respect to the action of $\mathbb{Z}/2 \cong \begin{bmatrix} \pm 1 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{bmatrix}$.

Proof. Define

$$G = \left\{ \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \mid A \in \mathrm{GL}_2(R), B \in \mathrm{GL}_{n-2}(R), \det(A)\det(B) = 1 \right\} \subseteq \mathrm{SL}_n(R)$$

to be the stabilizer of the standard copy of R^2 in R^n . Then

$$\begin{aligned}
& \left(\bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}(R^n/V) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(V)) \right)_{\text{SL}_n(R)} \\
& \cong \mathbb{Z} \otimes_{\mathbb{Z}[\text{SL}_n(R)]} \left(\bigoplus_{\substack{V \subseteq R^n \\ \text{rank}(V)=2}} \text{St}(R^n/V) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(V)) \right) \\
& \cong \mathbb{Z} \otimes_{\mathbb{Z}[\text{SL}_n(R)]} \left(\mathbb{Z}[\text{SL}_n(R)] \otimes_G \left(\text{St}(R^n/R^2) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2)) \right) \right) \\
& \cong \mathbb{Z} \otimes_G \left(\text{St}(R^n/R^2) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2)) \right).
\end{aligned}$$

Observe that the subgroup $\begin{bmatrix} \text{Id}_2 & * \\ 0 & \text{Id}_{n-2} \end{bmatrix} \subseteq G$ acts trivially, so the action by G factors through an action of

$$H = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \mid A \in \text{GL}_2(R), B \in \text{GL}_{n-2}(R), \det(A)\det(B) = 1 \right\} \cong (\text{SL}_2(R) \times \text{SL}_{n-2}(R)) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Thus

$$\begin{aligned}
& \cong \mathbb{Z} \otimes_G \left(\text{St}(R^n/R^2) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2)) \right) \\
& \cong \left(\text{St}(R^n/R^2) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2)) \right)_H \\
& \cong \left(\left(\text{St}_{n-2}(R^n/R^2) \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2)) \right)_{\text{SL}_2(R) \times \text{SL}_{n-2}(R)} \right)_{H/(\text{SL}_2(R) \times \text{SL}_{n-2}(R))} \\
& \cong \left(\text{St}_{n-2}(R^n/R^2)_{\text{SL}_2(R)} \otimes_{\mathbb{Z}} \tilde{H}_0(\mathbf{B}(R^2))_{\text{SL}_{n-2}(R)} \right)_{\mathbb{Z}/2} \\
& \cong \text{St}_{n-2}(R^n/R^2)_{\text{SL}_2(R)} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \tilde{H}_0(\mathbf{B}(R^2))_{\text{SL}_{n-2}(R)}
\end{aligned}$$

as claimed. \square

By Borel-Serre duality, the following is equivalent to [Theorem 1.3](#).

Proposition 5.9. *For all n , we have:*

$$\dim_{\mathbb{Q}} H_0(\text{SL}_{2n}(\mathcal{O}_d); \mathbb{Q} \otimes \text{St}_{2n}(\mathbb{Q}(\sqrt{d}))) \geq \begin{cases} 1 & \text{for } d = -43 \\ 2^n & \text{for } d = -67 \\ 6^n & \text{for } d = -163 \end{cases}$$

Proof. Recall that the coinvariants $\mathbb{Q} \otimes_{\mathbb{Z}} (\text{St}_n(\mathbb{Q}(\sqrt{d})))_{\text{SL}_n(\mathcal{O}_d)}$ are a representation of $\mathbb{Z}/2 \cong \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \text{Id}_{n-1} \end{bmatrix} \right\}$. Let t_n denote the multiplicity of the trivial representation, and s_n denote the multiplicity of the sign representation in $\mathbb{Q} \otimes_{\mathbb{Z}} (\text{St}_n(\mathbb{Q}(\sqrt{d})))_{\text{SL}_n(\mathcal{O}_d)}$. By [Lemma 5.6](#), [Lemma 5.7](#) and [Lemma 5.8](#), there is a surjection

$$(\text{St}_n)_{\text{SL}_n} \otimes \mathbb{Q} \twoheadrightarrow \left((\text{St}_{n-2})_{\text{SL}_{n-2}} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} (\text{St}_2)_{\text{SL}_2} \right) \otimes \mathbb{Q}$$

and so

$$t_n \geq t_{n-2}t_2 \quad \text{and} \quad s_n \geq s_{n-2}s_2.$$

Thus,

$$\dim_{\mathbb{Q}} H_0(\text{SL}_{2n}(\mathcal{O}_d); \mathbb{Q} \otimes \text{St}_{2n}(\mathbb{Q}(\sqrt{d}))) = t_{2n} + s_{2n} \geq (t_2)^n + (s_2)^n.$$

Since

$$t_n = \dim_{\mathbb{Q}} H_0(\mathrm{GL}_n(\mathcal{O}_d); \mathbb{Q} \otimes \mathrm{St}_n(\mathbb{Q}(\sqrt{d}))) \text{ and } t_n + s_n = \dim_{\mathbb{Q}} H_0(\mathrm{SL}_n(\mathcal{O}_d); \mathbb{Q} \otimes \mathrm{St}_n(\mathbb{Q}(\sqrt{d}))),$$

Theorem 5.1 and Corollary 5.5 give $t_2 = 0$ and

$$s_2 = \begin{cases} 1 & \text{for } d = -43 \\ 2 & \text{for } d = -67 \\ 6 & \text{for } d = -163. \end{cases}$$

□

Remark 5.10. Since $s_1 = 0$, one cannot easily use the proof strategy of Theorem 1.3 to show nonvanishing of $H^{\nu_n}(\mathrm{SL}_n(\mathcal{O}_d); \mathbb{Q})$ for all n odd. However, if one could show $s_3 > 0$, then these techniques would establish nonvanishing for n odd.

REFERENCES

- [AGM11] Avner Ash, Paul E. Gunnells, and Mark McConnell, *Torsion in the cohomology of congruence subgroups of $\mathrm{SL}(4, \mathbb{Z})$ and Galois representations*, J. Algebra **325** (2011), 404–415. MR 2745546 (2012b:11084) [9](#)
- [AR79] Avner Ash and Lee Rudolph, *The modular symbol and continued fractions in higher dimensions*, Invent. Math. **55** (1979), no. 3, 241–250. MR 553998 [2](#)
- [Avi00] David Avis, *A revised implementation of the reverse search vertex enumeration algorithm*, pp. 177–198, Birkhäuser Basel, Basel, 2000. [10](#)
- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478 [10](#)
- [BS73] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. MR 0387495 [1](#)
- [CFP15] Thomas Church, Benson Farb, and Andrew Putman, *Integrality in the Steinberg module and the top-dimensional cohomology of $\mathrm{GL}_n(\mathcal{O}_K)$* , arXiv preprint arXiv:1501.01307 (2015). [2](#), [4](#), [8](#), [9](#), [12](#)
- [Cha87] Ruth Charney, *A generalization of a theorem of Vogtmann*, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985), vol. 44, 1987, pp. 107–125. MR 885099 [5](#)
- [Coh66] Paul M Cohn, *On the structure of the GL_2 of a ring*, Publications Mathématiques de l’Institut des Hautes Études Scientifiques **30** (1966), no. 1, 5–53. [4](#), [9](#)
- [CP17] Thomas Church and Andrew Putman, *The codimension-one cohomology of $\mathrm{SL}_n \mathbb{Z}$* , Geom. Topol. **21** (2017), no. 2, 999–1032. MR 3626596 [3](#)
- [DSGG⁺16] Mathieu Dutour Sikirić, Herbert Gangl, Paul E. Gunnells, Jonathan Hanke, Achill Schürmann, and Dan Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, J. Pure Appl. Algebra **220** (2016), no. 7, 2564–2589. MR 3457984 [9](#), [10](#)
- [EVGS13] Philippe Elbaz-Vincent, Herbert Gangl, and Christophe Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math. **245** (2013), 587–624. MR 3084439 [9](#), [10](#)
- [LS76] Ronnie Lee and R. H. Szczarba, *On the homology and cohomology of congruence subgroups*, Invent. Math. **33** (1976), no. 1, 15–53. MR 0422498 [9](#)
- [MPP] Jeremy Miller, Peter Patzt, and Andrew Putman, *The Lee-Szczarba conjecture*, in preparation. [9](#)
- [Qui73] Daniel Quillen, *Finite generation of the groups k_i of rings of algebraic integers*, Higher K-Theories, Springer, 1973, pp. 179–198. [4](#)
- [Qui78] ———, *Homotopy properties of the poset of nontrivial p -subgroups of a group*, Advances in Mathematics **28** (1978), no. 2, 101–128. [5](#)
- [Rah13] Alexander Rahm, *The homological torsion of PSL_2 of the imaginary quadratic integers*, Transactions of the American Mathematical Society **365** (2013), no. 3, 1603–1635. [2](#), [9](#)
- [Ser53] Jean-Pierre Serre, *Groupes d’homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294. MR 0059548 (15,548c) [9](#)
- [Ser80] Jean Pierre Serre, *Trees*, Springer, 1980. [4](#)
- [Sol69] Louis Solomon, *The Steinberg character of a finite group with BN -pair*, Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968), 1969, pp. 213–221. [4](#)
- [Sou00] C. Soulé, *On the 3-torsion in $K_4(\mathbb{Z})$* , Topology **39** (2000), no. 2, 259–265. MR 1722032 (2000i:19008) [9](#)
- [vdK80] Wilberd van der Kallen, *Homology stability for linear groups*, Invent. Math. **60** (1980), no. 3, 269–295. MR 586429 (82c:18011) [4](#)
- [Vog85] Karen Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419. MR 799670 [9](#)
- [Wei73] Peter J. Weinberger, *On Euclidean rings of algebraic integers*, 321–332. MR 0337902 [2](#)

- [Yas10] Dan Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 6197, Springer, Berlin, 2010, pp. 385–396. MR 2721434 (2012g:11069) [10](#)

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