A TABLE OF ELLIPTIC CURVES OVER THE CUBIC FIELD OF DISCRIMINANT -23

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ABSTRACT. Let F be the cubic field of discriminant -23 and $\mathcal{O}_F$ its ring of integers. Let Γ be the arithmetic group $\mathrm{GL}_2(\mathcal{O}_F)$, and for any ideal $\mathfrak{n} \subset \mathcal{O}_F$ let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of level $\mathfrak{n}$. In [17], two of us (PG and DY) computed the cohomology of various $\Gamma_0(\mathfrak{n})$, along with the action of the Hecke operators. The goal of [17] was to test the modularity of elliptic curves over F. In the present paper, we complement and extend the results of [17] in two ways. First, we tabulate more elliptic curves than were found in [17] by using various heuristics ("old and new" cohomology classes, dimensions of Eisenstein subspaces) to predict the existence of elliptic curves of various conductors, and then by using more sophisticated search techniques (for instance, torsion subgroups, twisting, and the Cremona–Lingham algorithm) to find them. We then compute further invariants of these curves, such as their rank and representatives of all isogeny classes. Our enumeration includes conjecturally the first elliptic curves of ranks 1 and 2 over this field, which occur at levels of norm 719 and 9173 respectively.

1. Introduction

1.1. Let F be the cubic field of discriminant -23 and let $\mathcal{O}_F$ be its ring of integers. Let $\mathbf{G}$ be the reductive $\mathbb{Q}$-group $\mathrm{R}_{F/\mathbb{Q}}(\mathrm{GL}_2)$, let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be the arithmetic group $\mathrm{GL}_2(\mathcal{O}_F)$, and for any ideal $\mathfrak{n} \subset \mathcal{O}_F$ let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of level $\mathfrak{n}$. In [17] two of us (PG and DY) investigated the modularity of elliptic curves over F. In particular, for all ideals $\mathfrak{n}$ of norm up to some bound, we computed the action of the Hecke operators on the cohomology of the congruence subgroup $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathcal{O}_F)$ and identified classes with integral eigenvalues that are apparently attached to cuspidal automorphic forms on GL_2/F. For each such class ξ, we found an elliptic curve E/F of conductor $\mathfrak{n}$ such that $a_\mathfrak{p}(E) = a_\mathfrak{p}(\xi)$ for all primes $\mathfrak{p} \nmid \mathfrak{n}$ that we could check. Here $a_\mathfrak{p}(\xi)$ denotes the eigenvalue of the Hecke operator $T_\mathfrak{p}$ on ξ, and $a_\mathfrak{p}(E)$ comes from counting the points on E over the residue field $\mathbb{F}_\mathfrak{p} = \mathcal{O}_F/\mathfrak{p}$:

$$a_\mathfrak{p}(E) = \mathrm{N}(\mathfrak{p}) + 1 - \#E(\mathbb{F}_\mathfrak{p}).$$

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1.2. In this paper, we complement and extend the results of [17] in two ways. First, we investigate more fully the elliptic curves found in [17] by computing invariants such as their torsion subgroups and Mordell–Weil ranks. We also find representatives of the different isomorphism classes of curves within an isogeny class.

Second, we extend our table of curves through a variety of heuristics inspired by results in [17]. For instance, we use a heuristic of "old and new" cohomology classes and observations about the dimensions of Eisenstein subspaces in cohomology to make predictions about the dimensions of cuspidal subspaces. For many levels this prediction gives a one-dimensional cuspidal space, which then gives a prediction for the existence of an elliptic curve. In all such cases our searches yielded an apparently unique isogeny class of elliptic curves over F of that conductor. For other levels our heuristics predict cuspidal subspaces of dimension > 1. For some of these levels we found multiple isogeny classes of curves; for others we find no elliptic curves. We remark that most of these computations involve levels whose norms are far beyond those of levels where Hecke operator computations as in [17] are feasible. Thus we have no way of checking the "modularity" these curves, or even that the cohomology classes themselves appear to be attached to Galois representations. Nevertheless, in our opinion the fact that cohomology predicts the existence of these curves merely through dimension counts is compelling.[1]

Our paper fits into the long tradition of elliptic curve enumeration, the modern era of which began with Cremona's extensive tables of curves over $\mathbb{Q}$ [10] and imaginary quadratic fields [9]. Cremona's work has inspired many other efforts, including further work over $\mathbb{Q}$ [2, 28], as well as enumeration over $\mathbb{Q}(\sqrt{5})$ [4] and $\mathbb{Q}(e^{2\pi i/5})$ [16].

1.3. We now give an overview of the contents of this paper. In Section 2 we recall the setup from [17] and explain how we computed cohomology. We also describe the main heuristics that allow us to extend our computations far beyond that of [17]. In Section 3 we present various methods for attempting to find an elliptic curve over F of a given conductor. In Section 4, we address how to find all curves that are isogenous to a given curve E defined over F via an isogeny defined over F. In Section 5 we state our results and give tables that summarize various information about our dataset of elliptic curves. Finally, in Appendix A we give a small table of elliptic curves over F of conductor norm < 1187, along with some of their most important invariants; we believe this table gives a complete enumeration of isomorphism classes up to this bound. The full dataset we computed, which includes curves with conductors of norm up to approximately 20000 (with fairly complete data for curves of norm

[1]We note that recent remarkable work of P. Scholze [26] explains how to attach Galois representations to Hecke eigenclasses in the mod p and characteristic 0 cohomology of certain locally symmetric spaces. At present the example we consider falls outside the scope of this work, since our field F is neither totally real nor CM.

conductor less that 11575), is available online through the *L-functions and modular forms database* (`lmfdb.org` [29]).

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2. Cohomological automorphic forms and further heuristics

2.1. Throughout this paper we write F for the cubic field $\mathbb{Q}[x]/(x^3-x^2+1)$ of discriminant -23. We let a be a fixed root of x^3-x^2+1. The ring of integers $\mathcal{O}_F$ is then $\mathbb{Z}[a]$, and the unit group is generated by -1 and a.

In this section, we recall the setup of [17]. As above $\Gamma_0(\mathfrak{n})$ is a congruence subgroup of $\Gamma = \mathrm{GL}_2(\mathcal{O}_F)$. Instead of trying to work directly with automorphic forms on $\mathbf{G}$, we compute the cohomology of $\Gamma_0(\mathfrak{n})$; by a theorem of Franke [15] this allows us to work with certain automorphic forms over F, including those that should be attached to elliptic curves. Let C be the positivity domain of positive-definite binary quadratic forms over F, as constructed by Koecher (cf.[17, §3] and [22, §9]). The group Γ acts on C, and induces an action on C mod homotheties, which can be identified with the global symmetric space for $G = \mathbf{G}(\mathbb{R}) \simeq \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{C})$. More precisely, let $K \simeq \mathrm{O}(2) \times \mathrm{U}(2)$ be a maximal compact subgroup of G and let A_G be the split component. Then we have an isomorphism

$$C/\mathbb{R}_{>0} \simeq G/A_G K \simeq \mathfrak{H} \times \mathfrak{H}_3 \times \mathbb{R}, \tag{2.1}$$

where $\mathfrak{H}$ (respectively, $\mathfrak{H}_3$) is the hyperbolic plane (resp., hyperbolic 3-space). The explicit reduction theory due to Koecher enables us to construct a Γ-equivariant decomposition of C into polyhedral cones that induces a Γ-equivariant decomposition of $C/\mathbb{R}_{>0}$ into cells. The homology of the associated chain complex over $\mathbb{C}$ mod $\Gamma_0(\mathfrak{n})$ can be identified with $H^*(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}})$; here $\tilde{\Omega}_{\mathbb{C}}$ is the system of local coefficients attached to $\Omega \otimes \mathbb{C}$, where Ω is the *orientation module* of Γ.[2]

2.2. Over F, we have two sets of cohomological data on automorphic forms. First, we have computed the cohomology spaces $H^4(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}})$ and Hecke operators on levels up to norm 911; we then expect the cuspidal eigenclasses with rational eigenvalues to correspond to elliptic curves over F. Second, for many levels of norm higher than 911, including all of the levels of norm less than or equal to 11569, we have computed the spaces $H^4(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}})$ but *no* Hecke operators.[3] This means we cannot predict with certainty which ideals should be conductors of elliptic curves.

[2]We take this time to point out an error in [17], in which we neglected to include the orientation module in our coefficients. None of the results there or here are affected by this oversight.

[3]The Hecke computations became impractical at these levels because of our implementation. With better code we could undoubtedly treat some levels above norm 911, but even with this we do not expect to handle level norms above 5000.

Nevertheless, all is not lost. To extend our computations beyond level norm 911, we apply two heuristics derived from examination of the Hecke data where we can compute Hecke operators. The first concerns the size of the Eisenstein subspace of the cohomology, and the second concerns lifts of cohomology classes from lower levels to higher.

2.3. First, the Eisenstein cohomology is the cohomology that "comes from the boundary," and that should be eliminated from consideration when one wants to predict the size of the cuspidal subspace. For more details about Eisenstein cohomology, we refer to [19]; here we only recall the definition. Let $X = G/A_G K$ be the global symmetric space (2.1), and let X^{BS} be the partial compactification constructed by Borel and Serre [5]. The quotient $Y := \Gamma_0(\mathfrak{n})\backslash X$ is an orbifold, and the quotient $Y^{\mathrm{BS}} := \Gamma_0(\mathfrak{n})\backslash X^{\mathrm{BS}}$ is a compact orbifold with corners. The local system can be extended to the boundary, and we have

$$H^*(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}}) \simeq H^*(Y; \tilde{\Omega}_{\mathbb{C}}) \simeq H^*(Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}}),$$

where we have abused notation by denoting the original local system and its extension by the same symbol.

Now let $\partial Y^{\mathrm{BS}} = Y^{\mathrm{BS}} \smallsetminus Y$. The Hecke operators act on the cohomology of the boundary $H^*(\partial Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}})$, and the inclusion of the boundary $\iota\colon \partial Y^{\mathrm{BS}} \to Y^{\mathrm{BS}}$ induces a map on cohomology $\iota^*\colon H^*(Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}}) \to H^*(\partial Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}})$ compatible with the Hecke action. The kernel $H^*_!(Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}})$ of ι^* is called the *interior cohomology*; it equals the image of the cohomology with compact supports. The goal of Eisenstein cohomology is to use Eisenstein series and cohomology classes on the boundary to construct a Hecke-equivariant section $s\colon H^*(\partial Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}}) \to H^*(Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}})$ mapping onto a complement $H^*_{\mathrm{Eis}}(Y^{\mathrm{BS}}; \tilde{\Omega}_{\mathbb{C}})$ of the interior cohomology in the full cohomology. The image of s is called the *Eisenstein cohomology.* Computations from [17] suggest the following:

Heuristic 2.1. *The Eisenstein subspace of $H^4(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}})$ is rank $2c(\mathfrak{n}) - 1$, where $c(\mathfrak{n})$ is the number of $\Gamma_0(n)$-orbits on $\mathbb{P}^1(F)$.*

We remark that in principle we should be able to apply results of Harder [18] to explicitly determine this subspace. However, in practice it is easier to compute the Hecke operators on H^4 and to determine how large the space is from the Hecke eigenvalues (one looks for classes on which $T_{\mathfrak{p}}$ acts with eigenvalue $\mathrm{N}(\mathfrak{p}) + 1$.)

2.4. The second heuristic concerns how cuspidal eigenclasses at one level can appear at another. The data suggests that some of the same considerations in the Atkin–Lehner theory of oldforms [1] apply in cohomology. Recall that this theory is based on the observation that if $f(z)$ is a holomorphic weight k cuspform on $\Gamma_0(m) \subset \mathrm{SL}_2(\mathbb{Z})$, then $f(dz)$ is a holomorphic weight k cuspform on $\Gamma_0(m')$ for any m' divisible by m, where d is any divisor of m'/m. We observe the same in cohomology, which leads to the following prediction:

Heuristic 2.2. *Let ξ be a cuspidal Hecke eigenclass at level $\mathfrak{n} \subset \mathcal{O}_F$, and let $\mathfrak{N} \subset \mathcal{O}_F$ be divisible by $\mathfrak{n}$. Then for every proper, nontrivial divisor $\mathfrak{d}$ of $\mathfrak{N}/\mathfrak{n}$, there is a cuspidal Hecke eigenclass $\xi_{\mathfrak{d}}$ in the cohomology at level $\mathfrak{N}$ whose eigenvalues agree with those of ξ for $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid \mathfrak{N}$. Moreover, the classes $\xi_{\mathfrak{d}}$ are linearly independent in cohomology.*

We remark that this heuristic should follow from Casselman's generalization of Atkin–Lehner theory to automorphic representations of GL_2 [7,8]. However, we have not checked the details of this computation.

Example 2.3. Let $\mathfrak{p}_5$, $\mathfrak{p}_7$, and $\mathfrak{p}_{37}$ denote the degree 1 primes above 5, 7, and 37, respectively, and let $\mathfrak{N} = \mathfrak{p}_5\mathfrak{p}_7\mathfrak{p}_{37}$. The cohomology $H^4(\Gamma_0(\mathfrak{N}); \tilde{\Omega}_{\mathbb{C}})$ is 19-dimensional. Since F has class number one, [11, Theorem 7] implies that the number of boundary components in the Borel–Serre is

$$c(\mathfrak{N}) = \sum_{\mathfrak{d}|\mathfrak{N}} \phi_u(\mathfrak{d} + \mathfrak{N}\mathfrak{d}^{-1}),$$

where

$$\phi_u(\mathfrak{m}) = \#((\mathcal{O}_F/\mathfrak{m})^\times/\mathcal{O}_F^\times).$$

We compute that $\phi_u(\mathfrak{d} + \mathfrak{N}\mathfrak{d}^{-1}) = 1$ for each of the 8 divisors of $\mathfrak{N}$, and so $c(\mathfrak{N}) = 8$. Thus the expected cuspidal cohomology in $H^4(\Gamma_0(\mathfrak{N}); \tilde{\Omega}_{\mathbb{C}})$ is 4-dimensional. At level $\mathfrak{n} = \mathfrak{p}_5\mathfrak{p}_7$ we find a 1-dimensional cuspidal cohomology space and an elliptic curve of conductor $\mathfrak{n}$ to account for it. Since $\mathfrak{N}/\mathfrak{n} = \mathfrak{p}_{37}$ has two proper nontrivial divisors, Heuristic 2.2 tells us that we should expect a 2-dimensional contribution to the cohomology at level $\mathfrak{N}$. Similarly, the same happens at level $\mathfrak{n}' = \mathfrak{p}_5\mathfrak{p}_{37}$ which again produces a 2-dimensional contribution to the cohomology at level $\mathfrak{N}$. Therefore we expect (i) all the cuspidal eigenclasses at level $\mathfrak{N}$ are accounted for by cohomology for the levels $\mathfrak{n}, \mathfrak{n}'$, and (ii) no other levels dividing $\mathfrak{N}$ should have cuspidal cohomology. We find that this is true, and thus do not expect to find any elliptic curves over F of conductor $\mathfrak{N}$. Indeed, applying the techniques in Section 3 produced no curves over F of this conductor.

3. Strategies to find an elliptic curve

3.1. In this section, we describe various strategies for finding an elliptic curve E over F; some of these are described in [4] (for $F = \mathbb{Q}(\sqrt{5})$). There are different strategies to employ, depending on how much information one has about E. At the very least, one begins with an ideal $\mathfrak{n} \subset \mathcal{O}_F$ that one hopes is the conductor of an elliptic curve. If one is lucky, one has a list of the Hecke eigenvalues $a_{\mathfrak{p}}$ for a range of primes $\mathfrak{p}$ that are supposed to match the point counts of $E(\mathcal{O}_F/\mathfrak{p})$; such data opens the door to other techniques. However, it should be emphasized that, unlike the case of elliptic curves over $\mathbb{Q}$, even if one has complete explicit information about the automorphic form f on GL_2/F conjecturally attached to E, there is no direct way to construct an elliptic curve E_f with matching L-function. In other words, there is no known way to produce the period lattice $\Lambda \subset \mathbb{C}$ such that $E_f \simeq \mathbb{C}/\Lambda$. (For a discussion of these issues over real quadratic fields see [13]).

3.2. **Naive Enumeration.** The most naive strategy is to systematically loop through Weierstrass equations

$$E\colon y^2 + a_1xy + a_3 = x^3 + a_2x^2 + a_4x + a_6, \tag{3.1}$$

with $a_1, a_2, a_3, a_4, a_6 \in \mathcal{O}_F$ contained in some bounded subset of $\mathcal{O}_F$. For each elliptic curve E, we can compute the conductor $\mathfrak{n}_E$ to see if it matches the prediction from cohomology. If we have Hecke data, we can then check if it seems to agree with E.

This describes an algorithm that in principle will find all elliptic curves over F; however, it is of course of no use as soon as the curve with smallest Weierstrass coefficients in the target isogeny class has large coefficients in any equation. For example, enumerating all integral Weierstrass equations with two-digit coefficients over a cubic number field requires on the order of 200^{18} computations, which is infeasible. Most of the curves in our dataset could not be found with this technique. If one knows some $a_\mathfrak{p}$s, then gains can be made by sieving equations using congruence conditions imposed on the coefficients; still this is too inefficient to find curves with large Weierstrass coefficients.

3.3. **Torsion families.** We can refine the naive search in some cases if we can guess the torsion subgroup structure of E_f. If the torsion subgroup of E_f is one of the groups mentioned in Mazur's theorem or contains such a subgroup, we can use the parametrizations of [23] to significantly reduce our search area.

We use the following proposition to determine in which family to search:

Proposition 3.1. *Let ℓ be a prime in $\mathbb{Z}$, and E/F an elliptic curve. Then $\ell \mid \#E'(F)_{\mathrm{tors}}$ for some curve E' in the F-isogeny class of E if and only if for all odd primes $\mathfrak{p}$ at which E has good reduction $\ell \mid \mathrm{N}(\mathfrak{p}) + 1 - a_\mathfrak{p}$.*

Proof. One direction is easy. Suppose $\ell \mid \#E'(F)_{\mathrm{tors}}$. Then by the injectivity of the reduction map at primes of good reduction, $\ell \mid \#E'(\mathcal{O}_F/\mathfrak{p}) = \mathrm{N}(\mathfrak{p}) + 1 - a_\mathfrak{p}$. For the more difficult converse, see [20]. □

We can determine whether a curve in the isogeny class of E_f likely contains a F-rational ℓ-torsion point by applying Proposition 3.1 for all $a_\mathfrak{p}$ up to some bound on $\mathfrak{p}$. If this is the case, then we can search over the families of curves with ℓ-torsion for a curve in the isogeny class of E_f. Within a relatively small search space, we can find many curves with large coefficients much more quickly than with the naive search. For example, we found the curve

$$y^2 + a^2xy + a^2y = x^3 + (a+1)x^2 + (-200a^2 + 56a + 5)x - 739a^2 + 41a + 1139$$

with conductor (a^2-9) of norm 665 and the curve

$$\begin{aligned} y^2 + (a^2+1)xy + ay \\ &= x^3 + (-a^2+a+1)x^2 + (-249910a^2 + 438560a - 331055)x \\ &\quad + 86253321a^2 - 151364024a + 114261323 \end{aligned}$$

with conductor $(3a^2-14a+1)$ of norm 2065 by searching for curves with F-rational 6-torsion.[4]

3.4. **Twisting.** Recall that *quadratic twist* E'/F of an elliptic curve E/F is a curve that is isomorphic to E over a degree 2 extension of F. If we know an elliptic curve E/F of some conductor, we can compute quadratic twists to generate more curves over F, and under favorable conditions have information about the conductors of the twists. To make this precise, suppose the j-invariant $j(E)$ does not equal $0, 1728$. If E has Weierstrass equation

$$E\colon y^2 = x^3 + \alpha x + \beta, \quad \alpha, \beta \in F,$$

then for $d \in \mathcal{O}_F$ we define the d-twist E^d by

$$E^d\colon dy^2 = x^3 + \alpha x + \beta \tag{3.2}$$

We have the following well known proposition (for a proof, see [4]):

Proposition 3.2. *Let E/F be an elliptic curve with $j \neq 0, 1728$. If $\mathfrak{n}$ is the conductor of E and the ideal generated by $d \in \mathcal{O}_F$ is non-zero, square-free, and coprime to $\mathfrak{n}$, then the conductor of E^d is divisible by $d^2\mathfrak{n}$.*

Given E/F, we can use Proposition 3.2 to find the finite set of all d such that E^d might have norm conductor less than a given bound. We can then compute the quadratic twists by these d to find curves that may otherwise be difficult to find. For example, we found the curve

$$\begin{aligned} y^2 + (a+1)xy + (a^2+a+1)y \\ &= x^3 + (-a^2-a)x^2 + (-43a^2+63a-69)x - 198a^2 + 335a - 288 \end{aligned}$$

with conductor $(14a-3)$ and norm conductor 2645 using this method. This curve is a quadratic twist of $y^2 + axy + ay = x^3 + (a+1)x^2 + (6a-5)x + 4a^2 - 7a + 2$, which was found by searching over torsion families. Another example is

$$\begin{aligned} y^2 + (a^2+a)xy + a^2y \\ &= x^3 + (-a^2-a)x^2 + (-212a^2+305a-181)x - 1422a^2 + 2466a - 2087 \end{aligned}$$

[4]The given equation of the second curve is the canonical model, which is a global minimal model. The curve actually found using this method had the coefficients $[a_1, a_2, a_3, a_4, a_6] = [16a^2 + 24a + 10, -1872a^2 - 152a + 952, -1872a^2 - 152a + 952, 0, 0]$.

with conductor $(-15a^2+8a-1)$ and norm conductor 3025. This is a quadratic twist of $y^2+(4a^2+3a+1)xy+(4a^2+3a)y = x^3+(4a^2+3a)x^2$, which was again found by searching over torsion families.

3.5. **Curves with prescribed good reduction.** We also employ an algorithm due to Cremona–Lingham [12], which finds all elliptic curves with good reduction at primes outside of a finite set $\mathcal{S}$ of primes in a number field. A `Magma` [6] implementation of this algorithm was provided by Cremona. The algorithm has the advantage that it allows targeting a specific conductor. The drawback is that it can be difficult to use in practice, since a key step involves finding $\mathcal{S}$-integral points on elliptic curves.

Definition 3.3. The *m-Selmer groups* $F(\mathcal{S}, m)$ of F^* are defined to be

$$F(\mathcal{S}, m) = \{x \in F^*/(F^*)^m \mid \operatorname{ord}_{\mathfrak{p}}(x) \equiv 0 \bmod m \text{ for all } \mathfrak{p} \notin \mathcal{S}\},$$

where F^* is the multiplicative group of F.

Definition 3.4. $F(\mathcal{S}, m)_{mn}$ is defined to be the image of the natural map

$$F(\mathcal{S}, mn) \to F(\mathcal{S}, m).$$

The Cremona–Lingham algorithm computes the finite m-Selmer groups $F(\mathcal{S}, m)$ of F^* for $m = 2$, 3, 4, 6, and 12. From these it computes a finite set of possible j-invariants such that each elliptic curve with good reduction outside $\mathcal{S}$ has j-invariant in this set. These j-invariants are either $j = 0$ or 1728, cases which can be treated directly, or $\mathcal{S}$-integers in F satisfying

$$w \equiv j^2(j-1728)^3 \bmod F^{*6} \quad \text{for } w \in F(\mathcal{S}, 6)_{12}.$$

In the latter case j is of the form $j = x^3/w = 1728 + y^2/w$, where (x, y) is a $\mathcal{S}$-integral point on the elliptic curve $E_w : Y^2 = X^3 - 1728w$, of which there are finitely many by Siegel's Theorem. From this set of j-invariants, we construct each curve with the desired reduction properties (indeed, there are finitely many by Shafarevich's Theorem): for each $j = x^3/w$ (excluding $j = 0, 1728$, which are treated separately), we choose $u_0 \in F^*$ such that $(3u_0)^6 w \in F(\mathcal{S}, 12)$, and each curve is either of the form $E : Y^2 = X^3 - 3xu_0^2X - 2yu_0^3$ or is a quadratic twist $E^{(u)}$ for some $u \in F(\mathcal{S}, 2)$. We must also check that each curve found has good reduction at the primes above 2 and 3 (if these primes are not in $\mathcal{S}$).

The advantage of this approach is that it not only gives a way to find curves of given conductor, but also to prove there are no others. The disadvantage is that it is usually feasible to carry this through only for the smallest fields and conductors; the conductors in this paper are already too big. This is because, first of all, a large number of curves E_w must be considered individually. Worse still, for many w it is too hard to determine the set of all $\mathcal{S}$-integral points on E_w over F. The general method currently used for this involves first determining all *rational* points, i.e. determining the Mordell–Weil group $E_w(F)$. This is inherently very difficult. In particular, many of the groups $E_w(F)$ have rank 1 and are generated by a point of huge height (as predicted by the conjecture of Birch and Swinnerton-Dyer), and these generators are impossible to find with current techniques for curves over number fields. Ironically, in these hard cases there are never any $\mathcal{S}$-integral points in $E_w(F)$, because those points won't have huge height. So these hard cases are of no interest to us, but we can't prove it without knowing the generators!

Despite these difficulties, we used the Cremona–Lingham algorithm to find many curves with large coefficients, curves that would have been virtually impossible to find by the previous methods. Our implementations do not attempt to find all rational or $\mathcal{S}$-integral points but simply search, in natural search regions, for points in $E_w(F)$. For example, a search on E_w found the following curve defined over F with $\mathfrak{n} = (a^2 - 10a + 1)$ and norm conductor 865 using this method, which lies just outside the range of curves found in [17]:

$$y^2 + axy + (a^2+1)y = x^3 + (-a^2-1)x^2 + (-48a^2 + 85a - 63)x - 211a^2 + 368a - 277.$$

3.6. **A well-optimized search algorithm.** This section describes a more sophisticated algorithmic approach to using the ideas of the Cremona–Lingham method of the previous section. Again, we abandon the goal of proving completeness: our primary goal is to find all curves that actually exist. (Naturally one also wishes to prove non-existence of other curves, but this is simply too hard a problem with current algorithms.) Having adopted this attitude, in dealing with the large number of candidate curves E_w we are free to focus effort wherever we choose, and to switch between the candidates at will. Furthermore, we bring to bear some powerful techniques for searching for points on candidate curves. We have a two-pronged approach: the two main techniques described below complement each other to some extent (a point that is hard to find for one of them is not necessarily so hard for the other).

The program that performs all this is carefully written so as to minimize the effort required, starting with very quick searches on each candidate and gradually increasing the effort. It balances the running times of the different techniques, and focuses more effort on "more likely" candidates according to some theoretical heuristics. This program is implemented for general number fields, and is included in the Magma computational algebra system: the function is called `EllipticCurveSearch`.

The first main technique is a direct search for points on E_w which targets points especially likely to be of interest. This is based on a heuristic idea due to Elkies: if an elliptic curve has discriminant d and invariants c_4, c_6, then it is likely that for each archimedean absolute value v, $|c_4^3|_v, |c_6^2|_v$ and $|1728d|_v$ are all of roughly the same size. (If not, then there is a lot of cancellation in $c_4^3 - c_6^2 = 1728d$, and one expects this to occur not so frequently). Therefore we search for points on $E_w : y^2 = x^3 - 1728d$ by running over small values for x under a weighted norm that is determined by d. We also put in some non-archimedean information about c_4, so the search spaces consist of all $x \in F$ in the intersection of some $\mathbb{Z}$-module with some "box."

The second main technique is a tuned version of the generic approach to determining generators for the Mordell–Weil group of an elliptic curve over a number field, using the method of two-descent. Two-descent helps in two ways. First of all, it gives an upper bound on the rank of the Mordell–Weil group. In particular, when the bound is zero, or equals the rank of the group generated by points already found, we are done with E_w. Two-descent also gives a finite set of "two-covering curves" C with covering maps to $C \to E_w$, such that every point in $E_w(F)$ is the image of an F-rational point on some C. The advantage is that such a point has smaller height than its image on E_w, *if* one uses "nice" (i.e. minimized and reduced) models of the two-coverings. An algorithm for minimizing and reducing two-coverings over number fields is due to one of us (SD) and Fisher. Additionally, many two-coverings that have no rational points can be ruled out by computing Cassels–Tate pairings; an algorithm for this is due to one of us (SD).

All the above-mentioned algorithms have good implementations in `Magma`, so are available for use in our search for elliptic curves of given conductor. We explain how the search program works by describing what happens for some particular levels.

For level $\mathfrak{n} = (9a^2 - a - 15)$ of norm 2879, the space of forms has dimension 2, and there are two isogeny classes of elliptic curves. The search program has to individually consider 144 candidate curves E_w. We give details about the two values of w which yield the two curves.

For $w = a^2 - 24a - 17$, E_w has Mordell–Weil rank 3. Quick searches on E_w find two independent points; integral points in this rank 2 subgroup yield three elliptic curves, but none of conductor N. Using two-descent, a third independent point is quickly found (on the first two-cover chosen). Integral points in the full rank 3 group yield three more elliptic curves, including the curve with conductor N and discriminant w.

For $w = 17a^2 - 16a - 24$, E_w has Mordell–Weil rank 2. Quick searches on E_w find no rational points. Two-descent proves (first of all) that $\operatorname{rank} E_w(F) \leq 2$. In such cases, it is less likely that E_w has rank 2, than that it has rank 0 and that the two-coverings have no rational points, and indeed that a stronger condition holds, namely that Cassels–Tate pairings between distinct two-coverings are nontrivial. Therefore the program calculates the pairing, which turns out to be trivial in this case. Next, the program searches on reduced models of (two of the) two-coverings, obtaining two independent generators of $E_w(F)$. An $\mathcal{S}$-integral point in the group yields the second elliptic curve of conductor $\mathfrak{n}$ (and discriminant $a^6 w$).

The program spent a few seconds for each of these discriminants, mostly spent reducing the two-coverings. The entire process of finding the two curves of conductor $\mathfrak{n}$ took a minute or so. This involves some luck, in that the "right" values of d were among the first few discriminants for which the program chose to apply the harder techniques (two-descent etc). Some heuristics are used in this guesswork, aiming to test the more likely discriminants first, so it is a game of both strategy and luck.

For level $(-9a^2 - 11a + 3)$ of norm 2915, the space of forms has dimension 3 and there are three isogeny classes of elliptic curves. These were all found without using two-descent. There were 5184 candidate discriminants; the entire process took about five minutes. The curves found came from E_w of rank 3, 1 and 2 (in order of search effort required).

On the other hand, for many levels the space is not entirely composed of elliptic curves, and we do not have a good way to predict whether there should be elliptic curves. For such levels we must run the program, with some chosen setting of the "overall effort" parameter, on the full set of candidates E_w. A typical such level is $(12a^2 + 7a + 4)$ of norm 3325, where the space has dimension 3 and there is (apparently) only one isogeny class. It took several hours to process all 5184 candidate discriminants using all the techniques.

4. Enumerating the curves in an isogeny class

4.1. Now we turn to the next step in our table-building: given an elliptic curve E/F, we find representatives of all isomorphism classes of elliptic curves E'/F that are isogenous to E via an isogeny defined over F. Recall that two elliptic curves in an isogeny class are linked by a chain of prime degree isogenies; in particular, to enumerate an isogeny class we need to find all isogenies of prime degree, of which there are finitely many for curves that do not admit CM over the given number field. Over $\mathbb{Q}$, there is an algorithmic solution to this problem based on the following (see [10]):

(1) Mazur's theorem, which that states that if $\psi\colon E \to E'$ is a $\mathbb{Q}$-rational isogeny of prime degree, then $\deg\psi \leq 19$ or is in $\{37, 43, 67, 163\}$ [24].
(2) Vélu's formulas, which provide an explicit way to enumerate all prime degree isogenies with a given domain E (see [27, III Prop. 4.12] or [10, III Section 3.8]).

4.2. Vélu's formulas are valid for any number field and are implemented in `SAGE` and `Magma`, but there is currently no generalization of Mazur's theorem that gives us an explicit bound on the possible prime degree isogenies defined over a general number field.

Since we are interested in specific isogeny classes, we solve this problem by taking a less general perspective: we determine which prime degree isogenies are possible for a specific isogeny class using the following well-known result:

Theorem 4.1. *Let E be an elliptic curve over a number field K. For each prime number $\ell \in \mathbb{Z}$, let*

$$\rho_{E,\ell}\colon \operatorname{Gal}(\overline{\mathbb{Q}}/K) \to \operatorname{GL}(E[\ell]) \cong \operatorname{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$$

be the associated Galois representation on ℓ-torsion points, where $E[\ell]$ is the set (actually group) of ℓ-torsion points in $E(\overline{K})$. There exists an isogeny $E \to E'$ defined *over K of prime degree ℓ if and only if $\rho_{E,\ell}$ is reducible over $\mathbb{F}_\ell$. In particular, if $\rho_{E,\ell}$ is irreducible (over the algebraic closure of $\mathbb{F}_\ell$), then there can be no isogenies $E \to E'$ of prime degree ℓ.*

In what follows, we describe our implementation of an algorithm due to Billerey [3] that outputs a provably finite list of primes p such that a given elliptic curve E over a number field K might have a p-isogeny. We first develop the necessary background in Section 4.3, and then describe the implementation of algorithm in Section 4.4.

4.3. Let $M \subset \mathbb{Z}[X]$ be the subset of all monic polynomials that do not vanish at 0. For $P, Q \in M$, define $P * Q \in M$ by

$$(P * Q)(X) = \mathrm{Res}_Z(P(Z), Q(X/Z)Z^{\deg Q}), \tag{4.1}$$

where Res_Z is the resultant with respect to Z. This defines a commutative monoid structure on M with neutral element $\psi_1(X) = X - 1$ [3, Lemma 2.1]. For $r \geq 1$ and $P \in M$, define $P^{(r)} \in M$ by

$$P^{(r)}(X^r) = (P * \Psi_r)(X), \quad \text{where } \Psi_r(X) = X^r - 1. \tag{4.2}$$

Let K be a number field of odd degree d, and fix an elliptic curve E/K that does not admit CM over K. Let $\ell \in \mathbb{Z}$ be a prime number such that E has good reduction at every prime ideal of $\mathcal{O}_K$ dividing $\ell\mathcal{O}_K$. By abuse of language, we say that E has good reduction at ℓ. In this case, let

$$\ell\mathcal{O}_K = \prod_{\mathfrak{q}_i | \ell} \mathfrak{q}_i^{v_{\mathfrak{q}_i}(\ell)}$$

be the prime factorization of $\ell\mathcal{O}_K$. Associate to ℓ the polynomial

$$P_\ell^* = P_{\mathfrak{q}_1}^{(12 v_{\mathfrak{q}_1}(\ell))} * \cdots * P_{\mathfrak{q}_s}^{(12 v_{\mathfrak{q}_s}(\ell))},$$

where $P_\mathfrak{q}$ is defined as

$$P_\mathfrak{q}(X) = X^2 - a_\mathfrak{q} X + \mathrm{N}(\mathfrak{q}),$$

and where as usual $a_\mathfrak{q} = \mathrm{N}(\mathfrak{q}) + 1 - \#E(\mathcal{O}_K/\mathfrak{q})$. Then define the integer B_ℓ by

$$B_\ell = \prod_{k=0}^{[\frac{d}{2}]} P_\ell^*(\ell^{12k}).$$

where $[\frac{d}{2}]$ denotes the integer part of $\frac{d}{2}$. We have the following theorem of Billerey:

Theorem 4.2 ([3, Corollaire 2.5]). *Let $p \in \mathbb{Z}$ be a prime such that E admits a p-isogeny defined over K. Then one of the following is true:*

1. *the prime p divides $6\Delta_K N_{K/\mathbb{Q}}(\Delta_E)$; or*
2. *for all primes ℓ, the number B_ℓ is divisible by p (if $K = \mathbb{Q}$, we consider only $\ell \neq p$).*

Remark 4.3. The above criterion is effectively useful only if not all of the B_ℓ's are zero. This is the case for number fields of odd degree [3, Corollary 0.2]. We note that Billerey gives a similar criterion for the even degree case.

4.4. Let K be a number field of odd degree and E/K an elliptic curve without complex multiplication over K given by a Weierstrass equation with coefficients in $\mathcal{O}_K$. The following algorithm then outputs a provably finite set of primes containing $\mathrm{Red}(E/K)$, the set of primes p such that E has a p-isogeny (i.e., such that the Galois representation is reducible).

(1) Compute the set S_1 of prime divisors of $6\Delta_K N_{K/\mathbb{Q}}(\Delta_E)$.
(2) Let ℓ_0 be the smallest prime number not in S_1. The curve E has good reduction at ℓ_0. If $B_{\ell_0} \neq 0$, proceed to the next step. Otherwise, reiterate this step with the smallest prime number ℓ_1 not in S_1 and such that $\ell_1 > \ell_0$ etc. until we have some $B_\ell \neq 0$.
(3) We now have a non-zero integer B_ℓ. For greater efficiency, we can reiterate step 2 to obtain more such $B_\ell \neq 0$. We then define S_2 to be the set of prime factors of the greatest common divisor of the B_ℓ's we have obtained and define $S = S_1 \cup S_2$.
(4) The set S then contains $\mathrm{Red}(E/K)$, although it may contain other primes. We can eliminate some of these primes by calculating polynomials $P_\mathfrak{q}$ for some prime ideals $\mathfrak{q}$ of good reduction — in particular, if $P_\mathfrak{q}$ is irreducible modulo p (with $\mathfrak{q}$ not dividing p), then $p \notin \mathrm{Red}(E/K)$. The subset S' of S of prime numbers remaining is then usually small.

Now let K be our cubic number field F. Note that CM isogenies are defined over imaginary quadratic fields. Since F contains no such subfield, there are no CM isogenies defined over F. Therefore, by using this algorithm in combination with Vélu's formulas, we can find representatives of all isomorphisms in a given isogeny class of elliptic curves over F.

Example 4.4. Consider the curve E with Weierstrass coefficients $[a^2+1, -a^2+a-1, 0, 1, 0]$. The discriminant of E is $\Delta_E = 12a^2 - 25a - 43$, and

$$\mathrm{N}_{F/\mathbb{Q}}(\Delta_E) = -67375 = 5^3 \cdot 7^2 \cdot 11.$$

Thus $S_1 = \{2, 3, 5, 7, 11, 23\}$. Computing B_ℓ for $\ell \in \{13, 17, 19, 29\}$, we see that the greatest common divisor of the B_ℓ is $2^{16} \cdot 3^9$. Then $S_2 = \{2, 3\}$, and so $S = S_1 = \{2, 3, 5, 7, 11, 23\}$. Let $\mathfrak{p}_2$ denote the prime above 2. Then $P_{\mathfrak{p}_2}(x) = x^2 + 3x + 8$ is irreducible modulo 5, 7, and 11. Let $\mathfrak{p}_{17}$ denote the degree 1 prime above 17. Then $P_{\mathfrak{p}_{17}}(x) = x^2 + 6x + 17$ is irreducible modulo 23. It follows that $\mathrm{Red}(E/F) \subseteq \{2, 3\}$. Using Vélu's formulas, we compute 2 and 3-isogenies of E and all resulting curves until we get a set of elliptic curves which is closed under 2 and 3-isogenies, up to isomorphism. This computation yields a set of 12 representatives for the isomorphism

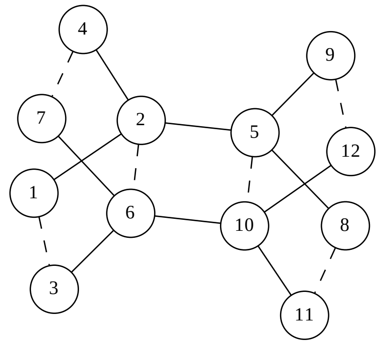

FIGURE 1. Prime isogeny graph for elliptic curve of norm conductor 385. The solid lines represent 2-isogenies, and the dashed lines represent 3-isogenies.

classes of elliptic curves in the isogeny class of E. This is the unique isogeny class of norm conductor 385 (label 140a). The prime isogeny graph is shown in Figure 1.

5. RESULTS & TABLES

5.1. We computed $H^4(\Gamma_0(\mathfrak{n}); \tilde{\Omega}_{\mathbb{C}})$ for the first 4246 levels, ordered by norm. This includes all of the levels of norm less than 11575 and three of the ideals of norm 11575. The current bottleneck preventing further computation performing linear algebra on large sparse matrices. Because of this, we expect to be able to push the computation further in special families, such as congruence subgroups of prime level.

Of these 4246 levels, Heuristic 2.1 implies that 1492 have nontrivial cuspidal cohomology. Of these, Heuristic 2.2 implies that 1175 have a nontrivial newspace. We found elliptic curves of matching conductor at 1020 of these levels, accounting for the full newspace in all but 213 levels. These elliptic curves comprise our dataset $\mathcal{D}$, of which we provide a sample in Appendix A. Of the remaining 213 levels, one falls within the range of our Hecke computations, and we can see that it corresponds to the base change of the classical weight two newform of level 23 with eigenvalues in $\mathbb{Q}(\sqrt{5})$ (cf. [17, §9]).

5.2. This leaves 212 levels with unexplained cuspidal cohomology. We note, however, that for each of these 212 levels, the cohomology that is left has rank 2 or greater; in particular there were no predicted new cuspidal subspaces of dimension 1 (according to our heuristics) for which we could not find a corresponding elliptic curve. This constitutes circumstantial evidence that our list of elliptic curves over F of norm conductor less than 11575 may be complete. That is to say, there is no clear reason (based on all the information now at hand) to predict another curve at any of these levels.

We judge it very likely that no elliptic curves are missing from our list. We conclude this on the basis of detailed information obtained from the search algorithms, and other circumstantial evidence. All the curves were found with a certain level of effort, and searching with substantially more effort produces no more curves. On close examination of the output, it seems likely that on the auxiliary curves E_w, all *integral* points, and all points of height small enough to be relevant, were found in the searches. If the conductors were substantially larger, one would be less confident; eventually there must certainly exist curves that would require much more effort to find using these methods. A curve that is missing would be likely to be an interesting curve with some unusual properties, such as large height.

In the course of computing the elliptic curves in $\mathcal{D}$, we encountered curves whose discriminant norm was small (less than 100000), but whose conductor lay outside the limits of our cohomology computations. These curves, together with the curves in $\mathcal{D}$, comprise a larger set of elliptic curves over F of small conductor. Partial data can be downloaded from [21] (as well as data for elliptic curves over other nonreal cubic fields). The complete larger dataset is available via the L-functions and Modular Forms Database (`http://www.lmfdb.org/`) [29].

5.3. In the remainder of this section, we provide tables summarizing our computations, and other highlights of the data. In all tables, only elliptic curves from $\mathcal{D}$ are included. In these tables, #isom refers to the number of isomorphism classes, #isog refers to the number of isogeny classes, $\mathfrak{n}$ and $\mathrm{N}(\mathfrak{n})$ refer respectively to the conductor and norm conductor of a given elliptic curve. We encode Weierstrass equations as vectors of coefficients: $[a_1, a_2, a_3, a_4, a_6]$.

Table 1 gives the number of isogeny classes and isomorphism classes in $\mathcal{D}$ that we found, sorted by algebraic rank. Note that, in a few cases, `Magma` gave an upper and lower bound on the rank that were not equal. In those instances, we switched to an isogenous curve and recomputed to try to get a larger lower bound or smaller upper bound. This was successful for every curve in our dataset. The first rank one elliptic curve we found occurs at norm conductor 719, and the first rank two curve occurs at norm conductor 9173. For every curve in $\mathcal{D}$, we found the algebraic rank agreed with the analytic rank, where analytic rank was computed by `Magma`. The algorithm used is heuristic, numerically computing derivatives of the L-function $L(E, s)$ at $s = 1$ until one appears to be nonzero.

In Table 2 we give the sizes of isogeny classes and the number of isogeny classes of each size in $\mathcal{D}$. We find some isogeny classes of cardinality 12, which is larger than the cardinalities observed over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$ (see [4]). The computation of one such class is described in Example 4.4; the other class appears in Appendix A at label 247a (norm conductor 665).

Table 3 gives the number of isogeny classes and the number of isomorphism classes with isogenies of each prime degree that we encountered. Note that these may not

TABLE 1. Elliptic curves over F

rank	**#isog**	**#isom**	**smallest** $N(\mathfrak{n})$
0	506	1729	89
1	812	1483	719
2	8	9	9173
total	1326	3221	

TABLE 2. Number of isogeny classes of a given size

size	1	2	3	4	6	8	10	12
number	645	634	64	484	82	70	1	2

TABLE 3. Prime isogeny degrees

degree	**#isog**	**#isom**	**example curve**	$N(\mathfrak{n})$
None	754	754	$[1, a^2+a-1, a^2+a, -a-1, -a^2+1]$	727
2	824	3844	$[a+1, -a^2-a-1, a^2+a, -a^2, -a^2+1]$	89
3	435	1452	$[a, a-1, 1, -a, 0]$	136
5	86	232	$[a, -a, a^2+a+1, -a, -2a^2+1]$	289
7	30	72	$[a, -a-1, a^2+1, 1, -a^2]$	625

represent all possible prime degrees of isogenies over F. We also provide an example curve, which need not have minimal norm conductor, that exhibits an isogeny of the given degree.

Table 4 gives the number of isomorphism classes of elliptic curves with given torsion structure. Again we include an example curve, which need not have minimal norm conductor, realizing a given torsion group. We find examples for all torsion subgroups that appear infinitely often over F, as proven in [25], and no others. It is unknown whether there are other subgroups that only appear finitely over F.

Finally, we consider whether any curves that we found have CM (i.e., whether or not $\mathrm{End}(E) \not\simeq \mathbb{Z}$). A complete list of CM j-invariants in F (computed using `Sage` code due to John Cremona and William Stein) is given in Table 5. Examining the j-invariants of the elliptic curves in $\mathcal{D}$, we see that no elliptic curve in $\mathcal{D}$ has CM. At larger levels we do see CM curves. In particular, we find some with CM in a quadratic order of discriminant -3, such as $[0, 0, a^2+a, 0, -a^2+1]$.

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TABLE 4. Torsion subgroups

torsion	**#isom**	**example curve**	N($\mathfrak{n}$)
0	738	$[-a^2+a,-a^2+a-1,-1,0,0]$	719
$\mathbb{Z}_2$	1222	$[-a^2+a,-a^2,a^2-a+1,-1,0]$	817
$\mathbb{Z}_3$	223	$[1,a,0,2a^2-a-3,2a^2-2a-3]$	773
$\mathbb{Z}_2\times\mathbb{Z}_2$	254	$[0,-a-1,0,6a-5,-4a^2+7a-3]$	512
$\mathbb{Z}_4$	301	$[a^2,-a^2-1,a^2,a^2+1,-a^2+a]$	911
$\mathbb{Z}_5$	53	$[-1,a^2-a,a,1,0]$	289
$\mathbb{Z}_6$	251	$[a,-a-1,a^2,-a^2+a+1,0]$	593
$\mathbb{Z}_7$	17	$[a^2-1,-a+1,a^2-a+1,0,0]$	293
$\mathbb{Z}_8$	29	$[a,1,a,0,0]$	553
$\mathbb{Z}_2\times\mathbb{Z}_4$	77	$[0,a^2+1,0,a^2,0]$	512
$\mathbb{Z}_9$	6	$[0,-a,-a-1,-a^2-a,0]$	107
$\mathbb{Z}_{10}$	20	$[a-1,-a^2-1,a^2-a,a^2,0]$	89
$\mathbb{Z}_{12}$	8	$[a,-a^2+a+1,a+1,0,0]$	185
$\mathbb{Z}_2\times\mathbb{Z}_6$	16	$[a,a+1,a,6a-5,4a^2-7a+2]$	115
$\mathbb{Z}_2\times\mathbb{Z}_8$	5	$[a,-1,a,-5a^2+8a-5,-4a^2+9a-4]$	805
$\mathbb{Z}_2\times\mathbb{Z}_{12}$	1	$[a^2,-a^2-a-1,a^2+1,-4a^2+11a-5,6a^2-15a+11]$	385

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TABLE 5. The CM j-invariants in F with fundamental discriminant D and conductor f.

D	f	j
-3	3	-12288000
-3	2	54000
-3	1	0
-4	2	287496
-4	1	1728
-7	2	16581375
-7	1	-3375
-8	1	8000
-11	1	-32768
-19	1	-884736
-43	1	-884736000
-67	1	-147197952000
-163	1	-262537412640768000
-23	2	$3792102031375a^2 - 6654675189750a + 5023465669375$
-23	1	$-1084125a^2 + 1904875a - 1437500$

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Appendix A. Table of Elliptic Curves over F

label	$\mathrm{N}(\mathfrak{n})$	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$33a$	89	$4a^2-a-5$	$[a+1,2a^2+2a+2,2a^2+a,8a^2+2a-3,6a^2-2a-5]$	0	$\mathbb{Z}_{10}$
			$[a+1,2a^2+2a+2,2a^2+a,3a^2+7a-8,2a^2-8a+3]$	0	$\mathbb{Z}_{10}$
			$[a+1,2a^2+2a+2,2a^2+a,-17a^2+72a-63,-144a^2+336a-291]$	0	$\mathbb{Z}_2$
			$[a+1,2a^2+2a+2,2a^2+a,-22a^2+82a-53,-88a^2+334a-321]$	0	$\mathbb{Z}_2$
$40a$	107	$-5a^2+3a$	$[0,2a^2+1,-a,4a^2+a,3a^2-2a-3]$	0	$\mathbb{Z}_9$
			$[0,2a^2+1,-a,14a^2+141a+100,-968a^2+444a+887]$	0	$\mathbb{Z}_3$
			$[0,2a^2+1,-a,34a^2+51a+20,-2515a^2+676a+1943]$	0	0
$43a$	115	$-2a^2-2a-3$	$[1,2a^2+4,-a^2-a,7a^2+4,4a^2-a+1]$	0	$\mathbb{Z}_{12}$
			$[1,2a^2+4,-a^2-a,-3a^2+15a-6,4a^2-4a+8]$	0	$\mathbb{Z}_2\times\mathbb{Z}_6$
			$[1,2a^2+4,-a^2-a,-28a^2+20a+9,-68a^2+67a+84]$	0	$\mathbb{Z}_6$
			$[1,2a^2+4,-a^2-a,-138a^2+250a-181,916a^2-1607a+1220]$	0	$\mathbb{Z}_6$
			$[1,2a^2+4,-a^2-a,17a^2-5a+9,19a^2+a+3]$	0	$\mathbb{Z}_4$
			$[1,2a^2+4,-a^2-a,-8a^2+40a-26,-69a^2+152a-113]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[1,2a^2+4,-a^2-a,-333a^2+550a-396,-4452a^2+7789a-5755]$	0	$\mathbb{Z}_2$
			$[1,2a^2+4,-a^2-a,-83a^2+250a-216,862a^2-1681a+1125]$	0	$\mathbb{Z}_2$
$52a$	136	$6a^2-2a-2$	$[a+1,7a^2+a+3,6a^2,36a^2-16a-20,6a^2-38a-26]$	0	$\mathbb{Z}_9$
			$[a+1,7a^2+a+3,6a^2+2a,190a^2-112a-180,-1128a^2-200a+500]$	0	$\mathbb{Z}_3$
			$[a+1,7a^2+a+3,6a^2,306a^2-156a-280,628a^2-852a-996]$	0	0
$58a$	161	$-5a^2+5a+4$	$[a^2+a,a+3,a-1,594a^2-4a-340,-3877a^2-3207a-212]$	0	$\mathbb{Z}_4$
			$[a^2+a,a+3,a-1,4a^2+a,2a^2-2a-3]$	0	$\mathbb{Z}_8$
			$[a^2+a,a+3,a-1,324a^2+131a-100,848a^2+2478a+1351]$	0	$\mathbb{Z}_2$
			$[a^2+a,a+3,a-1,39a^2+a-20,-28a^2-63a-32]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+a,a+3,a-1,44a^2+6a-20,-19a^2-23a-8]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+a,a+3,a-1,-156a^2-39a+60,-770a^2-64a+389]$	0	$\mathbb{Z}_2$
$59a$	167	$-5a^2+3a-3$	$[1,2a^2+4,-a^2-a+1,8a^2-a+4,6a^2-2a+1]$	0	$\mathbb{Z}_7$
			$[1,2a^2+4,-a^2-a+1,-12a^2+59a-21,-73a^2+227a-211]$	0	0

label	N(𝔫)	generator of 𝔫	Weierstrass model	rank	torsion
$70a$	185	$-a^2-5a+4$	$[a^2+a,-a^2+3a+3,-1,3a^2+5a+1,3a^2+a-1]$	0	$\mathbb{Z}_{12}$
			$[a^2+a,-a^2+3a+3,-1,3a^2+10a-89,7a^2-48a-401]$	0	$\mathbb{Z}_6$
			$[a^2+a,-a^2+3a+3,-1,-2347a^2+4145a-3209,-80439a^2+141063a-106939]$	0	$\mathbb{Z}_2$
			$[a^2+a,-a^2+3a+3,-1,18a^2+15a-29,-34a^2+9a-39]$	0	$\mathbb{Z}_4$
			$[a^2+a,-a^2+3a+3,-1,3a^2+5a-4,3a^2-3a-10]$	0	$\mathbb{Z}_2\times\mathbb{Z}_6$
			$[a^2+a,-a^2+3a+3,-1,-122a^2+260a-214,-1280a^2+2192a-1688]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+a,-a^2+3a+3,-1,3a^2+1,-a^2+6a-15]$	0	$\mathbb{Z}_{12}$
			$[a^2+a,-a^2+3a+3,-1,-137a^2+295a-179,-1445a^2+2353a-1473]$	0	$\mathbb{Z}_4$
$85a$	223	$-5a^2+3a-4$	$[a^2,2a^2+2a+3,-a-1,-5a^2+37a-52,53a^2-136a+94]$	0	$\mathbb{Z}_4$
			$[a^2,2a^2+2a+3,2a^2-a-1,9a^2+3a-1,8a^2-a-5]$	0	$\mathbb{Z}_8$
			$[a^2,2a^2+2a+3,-a-1,10a^2+2a-7,5a^2-9a-9]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2,2a^2+2a+3,-a-1,270a^2-598a-602,2135a^2-8720a-7783]$	0	$\mathbb{Z}_2$
			$[a^2,2a^2+2a+3,-a-1,25a^2-33a-42,9a^2-190a-148]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2,2a^2+2a+3,-a-1,20a^2-28a-42,19a^2-184a-169]$	0	$\mathbb{Z}_2$
$92a$	253	$7a^2-5a-5$	$[a^2+a,2a+3,a,179a^2-83a-170,1403a^2-497a-1172]$	0	$\mathbb{Z}_4$
			$[a^2+a,2a+3,a,4a^2+2a,4a^2-a-3]$	0	$\mathbb{Z}_8$
			$[a^2+a,2a+3,a,14a^2-3a-10,32a^2-18a-32]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+a,2a+3,a,9a^2-3a-10,33a^2-7a-28]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+a,2a+3,a,-36a^2-78a-90,142a^2+503a+127]$	0	$\mathbb{Z}_2$
			$[a^2+a,2a+3,a,-26a^2+72a+70,388a^2+107a-147]$	0	$\mathbb{Z}_2$
$94a$	259	$4a^2-7a-1$	$[0,2a^2+2a,-a^2-a,5a^2-3a-5,-4a^2-3a]$	0	$\mathbb{Z}_9$
			$[0,2a^2+2a,-a^2-a,1715a^2+1167a-5225,55166a^2+51300a-133449]$	0	0
			$[0,2a^2+2a,-a^2-a,-5a^2+17a+15,-24a^2+10a+21]$	0	$\mathbb{Z}_9$
			$[0,2a^2+2a,-a^2-a,205a^2-33a-205,-1334a^2-265a+374]$	0	$\mathbb{Z}_3$
$101a$	275	$8a^2-2a-3$	$[a^2+1,a^2+3a+2,-a,-81a^2-4a+40,-514a^2+290a+509]$	0	$\mathbb{Z}_2$
			$[a^2,4a^2+3a+1,a^2-a-1,18a^2-4a-13,3a^2-11a-10]$	0	$\mathbb{Z}_{10}$
			$[a^2+1,a^2+3a+2,-a,-96a^2+21a+25,-558a^2+346a+448]$	0	$\mathbb{Z}_2$
			$[a^2,4a^2+3a+1,a^2-a-1,-7a^2-9a-3,-118a^2+58a+111]$	0	$\mathbb{Z}_{10}$
			$[a^2+a,-a^2+2a+4,0,-3a-1,-8a^2-17a-8]$	0	$\mathbb{Z}_{10}$
			$[a^2+a,-a^2+2a+4,0,-25a^2-8a+9,39a^2-30a-45]$	0	$\mathbb{Z}_{10}$
$105a$	289	$3a^2-7a-2$	$[a^2,2a^2+3a+1,-a-1,9a^2-5,3a^2-3a-4]$	0	$\mathbb{Z}_5$
			$[a^2,2a^2+3a+1,-a-1,14a^2+15a-15,33a^2+45a-15]$	0	0
$107a$	293	$-5a^2-2a-2$	$[a^2+a,a^2+2a+5,a^2+a-1,9a^2+5a+4,12a^2+a-4]$	0	$\mathbb{Z}_7$
			$[a^2+a,a^2+2a+5,a^2+a-1,24a^2-5a-21,-46a^2-71a-48]$	0	0

label	N($\mathfrak{n}$)	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$128a$	344	$6a^2-2a-8$	$[a, 4a^2+2a, 2a^2+2a+2, -98128a^2+37792a+82728, 1108440a^2-10880182a-8872784]$	0	0
			$[a, 4a^2+2a, 2a^2+2a+2, 12a^2-8a-12, -10a^2-12a-4]$	0	$\mathbb{Z}_7$
			$[a, 4a^2+2a, 2a^2+2a+2, 42a^2+42a+8, 584a^2-432a-660]$	0	$\mathbb{Z}_7$
$132a$	359	$7a^2-6a-2$	$[1, 2a+3, -a^2-a, 3a^2+5a+3, 3a^2+2a]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, 18a^2+10a-2, 25a^2-21a-30]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, 53a^2-10a-37, 245a^2-31a-163]$	0	$\mathbb{Z}_2$
			$[1, 2a+3, -a^2-a, 53a^2-5a-37, 253a^2-46a-184]$	0	$\mathbb{Z}_2$
$140a$	385	$-6a^2+7a+5$	$[a^2+1, 2a^2+a+2, -a^2-a, 9a^2-a-3, 3a^2-2a-2]$	0	$\mathbb{Z}_{12}$
			$[a^2+1, 2a^2+a+2, -a^2-a, -11a^2+24a+27, 100a^2+79a-2]$	0	$\mathbb{Z}_4$
			$[a^2+1, 2a^2+a+2, -a^2-a, 9a^2-a-8, -a^2-5a-3]$	0	$\mathbb{Z}_2\times\mathbb{Z}_{12}$
			$[a^2+1, 2a^2+a+2, -a^2-a, 4a^2+14a-68, -22a^2-76a+148]$	0	$\mathbb{Z}_{12}$
			$[a^2+1, 2a^2+a+2, -a^2-a, 149a^2-346a-308, 508a^2-3446a-2909]$	0	$\mathbb{Z}_6$
			$[a^2+1, 2a^2+a+2, -a^2-a, 14a^2-16a-28, -16a^2-66a-58]$	0	$\mathbb{Z}_2\times\mathbb{Z}_6$
			$[a^2+1, 2a^2+a+2, -a^2-a, -5821a^2+6819a-3688, -141983a^2+262157a-249179]$	0	$\mathbb{Z}_2$
			$[a^2+1, 2a^2+a+2, -a^2-a, -26a^2+49a+7, 53a^2+168a-75]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+1, 2a^2+a+2, -a^2-a, -41a^2+74a-68, -220a^2+350a-467]$	0	$\mathbb{Z}_6$
			$[a^2+1, 2a^2+a+2, -a^2-a, -351a^2+429a-233, -2409a^2+4504a-4046]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+1, 2a^2+a+2, -a^2-a, 59a^2+69a-73, 307a^2+308a+124]$	0	$\mathbb{Z}_4$
			$[a^2+1, 2a^2+a+2, -a^2-a, -81a^2+119a-618, -2523a^2+775a-6857]$	0	$\mathbb{Z}_2$
$145a$	392	$-8a^2+6a+6$	$[a^2+1, 2a^2+2a, 2a, -1154a^2+2028a-1540, -27332a^2+47956a-36202]$	0	0
			$[1, 3a^2+a+2, 2a^2, 8a^2-4, 4a^2-2a-4]$	0	$\mathbb{Z}_7$
			$[a+1, 2a^2+a+4, 4a^2+2, 6a^2+2a+2, -4a^2+12a-8]$	0	$\mathbb{Z}_7$
$163a$	440	$8a^2+2a-6$	$[a^2, 4a^2+a+3, 2a^2+2a+2, 14a^2-4a-4, 4a^2-12a-8]$	0	$\mathbb{Z}_6$
			$[a^2, 4a^2+a+3, 2a^2+2a+2, 24a^2-4a-14, 32a^2-24a-42]$	0	$\mathbb{Z}_6$
			$[a^2, 4a^2+a+3, 2, 6a^2-16, -24a^2+28a-24]$	0	$\mathbb{Z}_2$
			$[a^2, 4a^2+a+3, 2, -74a^2+160a-136, -736a^2+1364a-1072]$	0	$\mathbb{Z}_2$
$168a$	449	a^2-8a	$[a^2+a, a^2+3a+3, a^2, 9a^2+4a-3, 8a^2-2a-7]$	0	$\mathbb{Z}_6$
			$[a^2+a, a^2+3a+3, a^2, 4a^2+9a-8, 5a^2-10a+2]$	0	$\mathbb{Z}_6$
			$[a^2+a, a^2+3a+3, a^2, -21a^2+59a-43, -120a^2+232a-186]$	0	$\mathbb{Z}_2$
			$[a^2+a, a^2+3a+3, a^2, -31a^2+59a-38, -133a^2+231a-174]$	0	$\mathbb{Z}_2$
$181a$	475	$-4a^2-7a$	$[0, 2a^2+2, -a, 5a^2-2a-1, a^2-2a-1]$	0	$\mathbb{Z}_5$
			$[0, 2a^2+2, -a, -5a^2+28a+19, 91a^2+34a-43]$	0	0

label	$\mathrm{N}(\mathfrak{n})$	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$185a$	503	a^2-a-8	$[a^2+a+1, a^2+4a+4, 3a^2+a-1, 15a^2+10a, 24a^2-2a-15]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, a^2+4a+4, 3a^2+a-1, -20a^2+30a+35, -50a^2+19a+43]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, a^2+4a+4, 3a^2+a-1, -10a^2-95a-65, -717a^2-670a-97]$	0	$\mathbb{Z}_2$
			$[a^2+a+1, a^2+4a+4, 3a^2+a-1, 5a^2-90a-70, -850a^2-696a-41]$	0	$\mathbb{Z}_2$
$186a$	505	$-8a+1$	$[a^2+a, 2a+5, -1, 6a^2+5a+4, 6a^2+2a-1]$	0	$\mathbb{Z}_6$
			$[a^2+a, 2a+5, -1, 11a^2-1, 5a^2-5a-4]$	0	$\mathbb{Z}_6$
			$[a^2+a, 2a+5, -1, -4a^2+20a-1, -14a^2+47a-22]$	0	$\mathbb{Z}_2$
			$[a^2+a, 2a+5, -1, 16a^2+15a-16, 40a^2+42a-57]$	0	$\mathbb{Z}_2$
$187a$	505	$-2a^2-7a+2$	$[a^2+a, a^2+3a+4, a^2-1, 12a^2+7a, 18a^2-10]$	0	$\mathbb{Z}_{10}$
			$[a^2+a, a^2+3a+4, a^2-1, -53a^2+67a-10, -277a^2+287a-39]$	0	$\mathbb{Z}_2$
			$[a^2+a, a^2+3a+4, a^2-1, 7a^2+2a, 2a^2-5a-4]$	0	$\mathbb{Z}_{10}$
			$[a^2+a, a^2+3a+4, a^2-1, 192a^2-13a-210, -768a^2-130a-74]$	0	$\mathbb{Z}_2$
$189a$	512	$8a^2-8$	$[-2a^2+2a, 4a^2+a-2, 4a+6, 8a^2-10a-10, -8a^2-12a-8]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[-2a^2+4a+2, 8a^2-6a-2, 8a^2+12a, -24a^2-24a-16, -144a^2+80a+16]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[-2a^2+2a, 4a^2-2a+4, -4a^2+8a+8, 16a^2-8a-8, 16a^2-32a-32]$	0	$\mathbb{Z}_8$
			$[2a+2, 4a-2, 4a^2+12a+4, -24a, -64a^2-16a]$	0	$\mathbb{Z}_4$
			$[-2a^2+4a+2, 8a^2+4, 8a^2+16a, 8a^2-8a-8, -128a^2+96a]$	0	$\mathbb{Z}_2$
			$[-2a^2+4a+2, 8a^2+4, 8a^2+16a+8, -104a^2+216a-216, -1760a^2+2880a-2240]$	0	$\mathbb{Z}_2$
$202a$	553	$9a^2-4a-2$	$[a, 3a^2+1, a^2-a-1, 11a^2-3a-88, -112a^2-9a+267]$	0	$\mathbb{Z}_4$
			$[a, 3a^2+1, a^2-a-1, 6a^2-3a-3, a^2-3a-2]$	0	$\mathbb{Z}_8$
			$[a, 3a^2+1, a^2-a-1, 6a^2-3a-8, -5a^2-3a+1]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a, 3a^2+1, a^2-a-1, a^2-3a-8, -22a^2+3a+7]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a, 3a^2+1, a^2-a-1, -44a^2-83a-3, -615a^2-105a+294]$	0	$\mathbb{Z}_2$
			$[a, 3a^2+1, a^2-a-1, -34a^2+77a-13, -177a^2+235a-156]$	0	$\mathbb{Z}_2$
$214a$	593	$8a^2-a-9$	$[a^2+1, a^2+2a+2, -1, 6a^2+a-1, 2a^2-a-2]$	0	$\mathbb{Z}_6$
			$[a^2+1, a^2+2a+2, -1, -4a^2+a+4, -a-1]$	0	$\mathbb{Z}_6$
			$[a^2+1, a^2+2a+2, -1, 26a^2+6a-11, 20a^2-42a-46]$	0	$\mathbb{Z}_2$
			$[a^2+1, a^2+2a+2, -1, 41a^2-9a-31, 121a^2-101a-148]$	0	$\mathbb{Z}_2$

label	N($\mathfrak{n}$)	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$217a$	595	$11a^2-4a-6$	$[a^2+a, a^2+2a+5, a^2, 143a^2-81a-146, 1071a^2-672a-1117]$	0	$\mathbb{Z}_8$
			$[a^2+a, a^2+2a+5, a^2, 8a^2+4a+4, 9a^2+a-2]$	0	$\mathbb{Z}_8$
			$[a^2+a, a^2+2a+5, a^2, 18a^2-a-6, 35a^2-25a-37]$	0	$\mathbb{Z}_2\times\mathbb{Z}_8$
			$[a^2+a, a^2+2a+5, a^2, -12177a^2+20079a+21134, -1762061a^2-394058a+693731]$	0	$\mathbb{Z}_2$
			$[a^2+a, a^2+2a+5, a^2, 53a^2-a-26, 3a^2-102a-77]$	0	$\mathbb{Z}_2\times\mathbb{Z}_8$
			$[a^2+a, a^2+2a+5, a^2, 708a^2-66a-446, -5205a^2-3555a+288]$	0	$\mathbb{Z}_8$
			$[a^2+a, a^2+2a+5, a^2, -42a^2+64a+74, -677a^2+103a+458]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+a, a^2+2a+5, a^2, -752a^2+1259a+1324, -27195a^2-3875a+12311]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+a, a^2+2a+5, a^2, -852a^2-91a+424, -12619a^2+6941a+12425]$	0	$\mathbb{Z}_4$
			$[a^2+a, a^2+2a+5, a^2, -687a^2+1559a+1514, -10301a^2-7264a+123]$	0	$\mathbb{Z}_2$
$233a$	625	$8a^2+3a+1$	$[a^2+a+1, a^2+4a+3, 3a^2+a-1, 14a^2+5a-4, 14a^2-8a-14]$	0	$\mathbb{Z}_5$
			$[a^2, 3a^2+2a+2, a^2-a, 5a^2+2a-2, a^2-1]$	0	$\mathbb{Z}_5$
			$[a^2, 3a^2+2a+2, a^2-a, -825a^2-158a+353, -12899a^2+5980a+11869]$	0	0
			$[a^2+a+1, a^2+4a+3, 3a^2+a-1, -36a^2+55a+61, 184a^2+287a+111]$	0	0
$243a$	649	$8a^2+a-8$	$[a^2+a+1, 2a+3, 3a^2-2, 3a^2+7a+2, 7a^2+2a-3]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 2a+3, 3a^2-2, 8a^2+2a+7, 16a^2-5a-2]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 2a+3, 3a^2-2, -32a^2+62a-38, -142a^2+257a-191]$	0	$\mathbb{Z}_2$
			$[a^2+a+1, 2a+3, 3a^2-2, -27a^2+57a-38, -153a^2+276a-221]$	0	$\mathbb{Z}_2$
$247a$	665	$9a^2+a-8$	$[1, 2a+3, -a^2-a, 8a+1, a^2+2a+2]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -25a^2+53a-34, 87a^2-147a+112]$	0	$\mathbb{Z}_2\times\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -440a^2+783a-584, 6190a^2-10854a+8194]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -10a^2+43a-44, 148a^2-196a+90]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -4005a^2+7013a-5269, -176412a^2+309544a-233665]$	0	$\mathbb{Z}_2$
			$[1, 2a+3, -a^2-a, -50a^2+93a-59, -269a^2+477a-347]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -8190a^2+8408a-1889, -159767a^2+214454a-309664]$	0	$\mathbb{Z}_2$
			$[1, 2a+3, -a^2-a, -4250a^2+7093a-5069, -176459a^2+308287a-234587]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[1, 2a+3, -a^2-a, -60a^2+108a-69, -189a^2+332a-238]$	0	$\mathbb{Z}_2\times\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -495a^2+803a-544, 6075a^2-10829a+8078]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, 215a^2-347a+246, -973a^2+1753a-1378]$	0	$\mathbb{Z}_6$
			$[1, 2a+3, -a^2-a, -4230a^2+7058a-5049, -178139a^2+311112a-236758]$	0	$\mathbb{Z}_2$

label	N($\mathfrak{n}$)	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$254a$	685	$-7a^2+5a-7$	$[a^2+1, a^2+a, -a^2-a, 179a^2-96a-169, 1188a^2-457a-1022]$	0	$\mathbb{Z}_4$
			$[a^2+1, a^2+a, -a^2-a, 4a^2-a-4, -a^2-a]$	0	$\mathbb{Z}_8$
			$[a^2+1, a^2+a, -a^2-a, 14a^2-6a-14, 16a^2-14a-20]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+1, a^2+a, -a^2-a, -2196a^2+2034a-1514, -20218a^2+56780a-54792]$	0	$\mathbb{Z}_2$
			$[a^2+1, a^2+a, -a^2-a, 9a^2+4a-19, 12a^2-3a-38]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+1, a^2+a, -a^2-a, -136a^2+129a-94, -357a^2+953a-812]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+1, a^2+a, -a^2-a, 74a^2+39a-24, 65a^2-335a-336]$	0	$\mathbb{Z}_4$
			$[a^2+1, a^2+a, -a^2-a, -396a^2+224a+126, -1212a^2+3150a+1332]$	0	$\mathbb{Z}_2$
$265a$	712	$6a^2-10a-8$	$[a+1, 7a^2+3a+5, 4a^2+2a+2, 60a^2-2a-24, 104a^2-60a-100]$	0	$\mathbb{Z}_6$
			$[a+1, 7a^2+3a+5, 4a^2+2a+2, 40a^2+18a-44, 72a^2-84a-72]$	0	$\mathbb{Z}_6$
			$[a+1, 7a^2+3a+5, 4a^2+4a+2, 14a^2-8a-4, -28a^2-4a+12]$	0	$\mathbb{Z}_2$
			$[a+1, 7a^2+3a+5, 4a^2+4a+2, -626a^2-8a+316, 3492a^2-2564a-3892]$	0	$\mathbb{Z}_2$
$266a$	719	a^2-a-9	$[a^2+a+1, 4a+3, 2a^2+a-2, 11a^2+8a, 17a^2-a-11]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 2a^2+a-2, 6a^2+13a, 14a^2+a-1]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 2a^2+a-2, 31a^2-7a-20, 84a^2-74a-104]$	0	$\mathbb{Z}_2$
			$[a^2+a+1, 4a+3, 2a^2+a-2, 26a^2-2a-25, 82a^2-68a-97]$	0	$\mathbb{Z}_2$
$268a$	719	$11a^2-4a-5$	$[a^2+1, 2a^2+2a+2, -a, 12a^2+a-5, 7a^2-7a-9]$	1	0
			$[a^2+1, 2a^2+2a+2, -a, 12a^2+a-5, 7a^2-7a-9]$	1	0
$269a$	721	$8a^2-9$	$[a^2+a+1, 4a+3, 3a^2+a-1, 8a^2+9a, 14a^2-7]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 3a^2+a-1, 13a^2+4a, 17a^2-9a-10]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 3a^2+a-1, -12a^2+34a-20, -55a^2+83a-55]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 3a^2+a-1, -1322a^2+2339a-1735, -33252a^2+58345a-44010]$	0	$\mathbb{Z}_2$
			$[a^2+a+1, 4a+3, 3a^2+a-1, -7a^2+29a-15, -74a^2+119a-83]$	0	$\mathbb{Z}_6$
			$[a^2+a+1, 4a+3, 3a^2+a-1, -1487a^2+2539a-1490, -35052a^2+58054a-43204]$	0	$\mathbb{Z}_2$
$270a$	727	$10a^2-7a-7$	$[a^2+a, a^2+2a+4, -1, 9a^2+4a-1, 8a^2-a-5]$	1	0
			$[a^2+a, a^2+2a+4, -1, 9a^2+4a-1, 8a^2-a-5]$	1	0
$283a$	773	$-3a^2+12a-5$	$[1, a+3, 1, 2a^2+a, 4a^2-2a-5]$	0	$\mathbb{Z}_3$
			$[1, a+3, 1, -33a^2+51a-40, -138a^2+250a-193]$	0	0

label	N($\mathfrak{n}$)	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$290a$	805	$3a^2-6a-10$	$[a^2, 2a^2+3a+3, -1, 12a^2+5a-1, 12a^2+a-6]$	0	$\mathbb{Z}_8$
			$[a^2, 2a^2+3a+3, -1, -5278a^2+9825a-7216, -281397a^2+497785a-377298]$	0	$\mathbb{Z}_2$
			$[a^2, 2a^2+3a+3, -1, 12a^2+5a-6, 3a^2-3a-10]$	0	$\mathbb{Z}_2\times\mathbb{Z}_8$
			$[a^2, 2a^2+3a+3, -1, -33a^2+10a-41, -326a^2+68a-28]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2, 2a^2+3a+3, -1, -388a^2+555a-461, -4638a^2+8031a-6106]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2, 2a^2+3a+3, -1, -398a^2-455a-181, -10630a^2-3651a+3018]$	0	$\mathbb{Z}_4$
			$[a^2, 2a^2+3a+3, -1, 57a^2-51, -64a^2-110a-28]$	0	$\mathbb{Z}_8$
			$[a^2, 2a^2+3a+3, -1, -1178a^2+5a-426, 11693a^2+9629a-14206]$	0	$\mathbb{Z}_2$
$291a$	808	$-6a^2-2a-4$	$[a, 6a^2+a+4, 2a^2-2, 28a^2-6a-6, 24a^2-16a-16]$	0	$\mathbb{Z}_5$
			$[a, 6a^2+a+4, 2a^2-2, -52a^2+164a-286, -1306a^2+1992a-2502]$	0	0
$294a$	809	$9a^2-9a-1$	$[a, 4a^2+a+2, a^2-1, 15a^2-5a-8, 7a^2-9a-9]$	0	$\mathbb{Z}_5$
			$[a, 4a^2+a+2, a^2-1, 120a^2+15a-53, -83a^2-503a-331]$	0	0
$294b$	809	$9a^2-9a-1$	$[a, 3a^2+a+2, 0, 10a^2+2a-1, 19a^2-3a-12]$	0	$\mathbb{Z}_6$
			$[a, 3a^2+a+2, 0, 25a^2-13a-21, 47a^2-40a-56]$	0	$\mathbb{Z}_6$
			$[a, 3a^2+a+2, 0, -110a^2+97a+139, -454a^2-295a+37]$	0	$\mathbb{Z}_2$
			$[a, 3a^2+a+2, 0, -130a^2+82a+139, -843a^2-113a+396]$	0	$\mathbb{Z}_2$
$297a$	817	$-a^2-7a-8$	$[a, 4a^2+a, 0, 9a^2-6a-9, -3a^2-5a-2]$	1	$\mathbb{Z}_2$
			$[a, 4a^2+a, 0, 9a^2-a-9, 4a^2-5a-5]$	1	$\mathbb{Z}_2$
$305a$	829	$6a^2-a-10$	$[0, 2, -a, 1, 0]$	1	0
			$[0, 2, -a, 1, 0]$	1	0
$315a$	851	$-2a^2+10a+1$	$[a^2, a+3, a^2-1, a^2+3a+2, a^2+2a-1]$	0	$\mathbb{Z}_4$
			$[a^2, 3a^2+a+3, -a, -138a^2+239a-194, -1411a^2+2382a-1788]$	0	$\mathbb{Z}_2$
			$[a^2, 3a^2+a+3, -a, 2a^2+14a-14, -25a^2+49a-44]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2, 3a^2+a+3, -a, -18a^2+29a+6, -47a^2+48a-24]$	0	$\mathbb{Z}_2$
$322a$	865	$9a^2-9a-8$	$[1, a^2+2, -a^2-a+1, 2, -a^2+1]$	0	$\mathbb{Z}_6$
			$[1, a^2+2, -a^2-a+1, -35a^2+22, 43a^2-49a-61]$	0	$\mathbb{Z}_6$
			$[1, a^2+2, -a^2-a+1, -45a^2-40a-3, -350a^2-64a+151]$	0	$\mathbb{Z}_2$
			$[1, a^2+2, -a^2-a+1, -55a^2-35a+7, -308a^2-124a+82]$	0	$\mathbb{Z}_2$
$322b$	865	$9a^2-9a-8$	$[a, 2a^2+2, a, a^2+4a-4, -5a^2+11a-10]$	0	$\mathbb{Z}_4$
			$[a, 2a^2+2, -a, -768a^2+1354a-1029, -15730a^2+27595a-20831]$	0	$\mathbb{Z}_2$
			$[a, 2a^2+2, -a, -43a^2+84a-64, -283a^2+500a-379]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a, 2a^2+2, -a, -38a^2+94a-59, -288a^2+521a-363]$	0	$\mathbb{Z}_4$

label	$\mathrm{N}(\mathfrak{n})$	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$325a$	875	$5a^2+5a+5$	$[a^2+1, 2a^2+2a, -a, 10a^2-a-7, 5a^2-4a-6]$	0	$\mathbb{Z}_8$
			$[a^2+1, 2a^2+2a, -a, 3340a^2-1201a-2807, -90789a^2-988a+50986]$	0	$\mathbb{Z}_4$
			$[a^2+1, 2a^2+2a, -a, 20a^2-6a-17, -20a^2-19a-3]$	0	$\mathbb{Z}_2\times\mathbb{Z}_8$
			$[a^2+1, 2a^2+2a, -a, 215a^2-76a-182, -1389a^2-238a+611]$	0	$\mathbb{Z}_2\times\mathbb{Z}_4$
			$[a^2+1, 2a^2+2a, -a, -15a^2-16a-12, -171a^2-40a+75]$	0	$\mathbb{Z}_8$
			$[a^2+1, 2a^2+2a, -a, 210a^2-71a-197, -1345a^2-304a+492]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[a^2+1, 2a^2+2a, -a, -340a^2+504a+203, -8180a^2+3461a+4497]$	0	$\mathbb{Z}_2$
			$[a^2+1, 2a^2+2a, -a, 680a^2-566a-837, 8966a^2-8093a-11269]$	0	$\mathbb{Z}_2$
$325b$	875	$5a^2+5a+5$	$[a^2+a, -a^2+3a+3, a^2, -2a^2+5a+5, -a^2+2a+2]$	0	$\mathbb{Z}_6$
			$[a^2+a, -a^2+3a+3, a^2, -57a^2+10a+40, 28a^2-94a-87]$	0	$\mathbb{Z}_6$
			$[a^2+a, -a^2+3a+3, a^2, 73a^2+385a-160, -4276a^2+11782a+8122]$	0	$\mathbb{Z}_2$
			$[a^2+a, -a^2+3a+3, a^2, 8a^2-40a-35, -39a^2-216a-141]$	0	$\mathbb{Z}_6$
			$[a^2+a, -a^2+3a+3, a^2, 203a^2-270a-345, -2047a^2+1971a+2603]$	0	$\mathbb{Z}_2$
			$[a^2+a, -a^2+3a+3, a^2, -22a^2-10a, 82a^2-365a-327]$	0	$\mathbb{Z}_6$
$333a$	883	$-7a^2+4a-7$	$[a^2+a+1, a^2+3a+2, 2a^2+a-2, 10a^2+4a-2, 12a^2-2a-9]$	1	0
			$[a^2+a+1, a^2+3a+2, 2a^2+a-2, 10a^2+4a-2, 12a^2-2a-9]$	1	0
$336a$	905	$-4a^2-7a+5$	$[a, 3a^2, -a, 6a^2-3a-5, -2a^2-4a-2]$	0	$\mathbb{Z}_{10}$
			$[a, 3a^2, -a, 11a^2+7a, a^2+11a+8]$	0	$\mathbb{Z}_{10}$
			$[a, 3a^2, -a, -179a^2-253a-115, -3512a^2-1301a+947]$	0	$\mathbb{Z}_2$
			$[a, 3a^2, -a, -174a^2-253a-115, -3604a^2-1300a+990]$	0	$\mathbb{Z}_2$
$338a$	911	$11a^2-7a-7$	$[1, a^2+a+4, -a^2-a, 266a^2-605a-603, 2756a^2-8935a-8313]$	0	$\mathbb{Z}_2$
			$[1, a^2+a+4, -a^2-a, 6a^2+2, 3a^2-5a-3]$	0	$\mathbb{Z}_4$
			$[1, a^2+a+4, -a^2-a, 21a^2-35a-33, 43a^2-192a-167]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[1, a^2+a+4, -a^2-a, 16a^2-25a-23, 70a^2-237a-217]$	0	$\mathbb{Z}_2$
$351a$	952	$10a^2+2a$	$[a^2+1, 4a^2+6a+5, 8a^2-2, -1344666a^2+2359820a-1781376, -1310904916a^2+2300477896a-1736579460]$	0	$\mathbb{Z}_2$
			$[a^2+1, 4a^2+6a+5, 4a^2-2, -22781142a^2+39978218a-30178668, -75113763134a^2+131815465490a-99504551060]$	0	$\mathbb{Z}_2$
$354a$	959	$a^2+4a-11$	$[a+1, 4a^2+a+2, a^2, 16a^2-3a-9, 8a^2-10a-12]$	1	$\mathbb{Z}_2$
			$[a+1, 4a^2+a+2, a^2, 21a^2-8a-9, 10a^2-22a-11]$	1	$\mathbb{Z}_2$
$363a$	991	$5a^2-4a-11$	$[1, a+3, -a+1, -a^2+3a+4, -3a^2+2a+3]$	1	$\mathbb{Z}_3$
			$[1, a+3, -a+1, 64a^2-7a-41, -112a^2-81a+2]$	1	0
$375a$	1003	$-8a^2+10a+3$	$[1, 4, 0, 6, a^2+a+3]$	1	0
			$[1, 4, 0, 6, a^2+a+3]$	1	0

label	N($\mathfrak{n}$)	generator of $\mathfrak{n}$	Weierstrass model	rank	torsion
$380a$	1033	$12a^2-a-3$	$[a,4a^2+2a+2,-1,18a^2-2a-11,9a^2-11a-14]$	1	$\mathbb{Z}_2$
			$[a,4a^2+2a+2,-1,13a^2+3a-16,-9a^2+6a-21]$	1	$\mathbb{Z}_2$
$383a$	1045	$13a^2-5a-7$	$[a^2+a,a^2+2a+3,a^2+a,5a^2+2a+1,4a^2-1]$	0	$\mathbb{Z}_{10}$
			$[a^2+a,a^2+2a+3,a^2+a,-10a^2+12a+11,-a^2+42a+32]$	0	$\mathbb{Z}_{10}$
			$[a^2+a,a^2+2a+3,a^2+a,-655a^2+502a-539,-3116a^2+10706a-8921]$	0	$\mathbb{Z}_2$
			$[a^2+a,a^2+2a+3,a^2+a,-35a^2+27a-39,-121a^2+158a-147]$	0	$\mathbb{Z}_2$
$389a$	1064	$6a^2+8a-2$	$[a^2+1,4a^2+2a+5,2a^2,26a^2-2a-4,22a^2-16a-16]$	0	$\mathbb{Z}_9$
			$[a^2+1,4a^2+2a+5,4a^2,54a^2+14a-8,172a^2-116a-176]$	0	$\mathbb{Z}_9$
			$[a^2+1,4a^2+2a+5,2a^2,-324a^2+268a+396,-2248a^2-1372a+236]$	0	$\mathbb{Z}_3$
			$[a^2+1,4a^2+2a+5,4a^2,534a^2-1276a-1758,-13880a^2-24544a-16012]$	0	0
$394a$	1080	$6a^2-6a-12$	$[a^2+a+1,6a^2+2a+7,6a^2+2a-2,56a^2-6a-8,92a^2-58a-66]$	0	$\mathbb{Z}_7$
			$[a^2+a+1,6a^2+2a+7,6a^2+2a-2,-184a^2+324a-358,-3268a^2+5492a-3856]$	0	0
$399a$	1097	$-5a^2+8a-13$	$[a,3a^2+a+2,-1,11a^2-2a-5,4a^2-6a-6]$	1	0
			$[a,3a^2+a+2,-1,11a^2-2a-5,4a^2-6a-6]$	1	0
$405a$	1111	$5a^2-a-11$	$[a,2a^2+2a,-a-1,6a^2-a-5,-3a-2]$	1	0
			$[a,2a^2+2a,-a-1,6a^2-a-5,-3a-2]$	1	0
$405b$	1111	$5a^2-a-11$	$[a+1,3a^2+a+3,a^2+a,-6252a^2+10261a-7932,-336552a^2+572642a-429068]$	0	0
			$[a+1,3a^2+a+3,a^2+a,13a^2+a-2,13a^2-3a-8]$	0	$\mathbb{Z}_5$
			$[a+1,3a^2+a+3,a^2+a,-7a^2+6a-17,-2a^2+65a-54]$	0	$\mathbb{Z}_5$
$406a$	1111	$a^2-3a-10$	$[a^2+a,-a^2+a+4,-1,-3020a^2+8537a-2126,-20301a^2+191896a-254758]$	0	0
			$[a^2+a,-a^2+a+4,-1,2a+4,-a^2+a+2]$	0	$\mathbb{Z}_5$
			$[a^2+a,-a^2+a+4,-1,-5a^2+7a-6,14a^2+8a-42]$	0	$\mathbb{Z}_5$
$421a$	1133	$12a^2-7a-7$	$[a^2,4a^2+3a+1,-1,18a^2-3a-15,6a^2-14a-13]$	1	$\mathbb{Z}_3$
			$[a^2,4a^2+3a+1,-1,18a^2-13a,a^2+6a+9]$	1	0
$425a$	1151	$-9a^2+5a-6$	$[a^2,3a^2+3a+1,a^2-a-1,14a^2-4a-11,-a^2-10a-7]$	1	0
			$[a^2,3a^2+3a+1,a^2-a-1,14a^2-4a-11,-a^2-10a-7]$	1	0
$426a$	1151	$-6a^2-5a-5$	$[a,3a^2+a+1,a^2-a-1,9a^2-2a-6,2a^2-5a-6]$	0	$\mathbb{Z}_3$
			$[a,3a^2+a+1,a^2-a-1,-16a^2+48a-51,-127a^2+225a-199]$	0	0
$435a$	1169	$-5a^2+a-8$	$[1,a^2+a+4,-a^2+1,-119a^2-101a-130,-1672a^2-892a-184]$	0	$\mathbb{Z}_2$
			$[1,a^2+a+4,-a^2+1,-14a^2+34a-20,32a^2-51a+40]$	0	$\mathbb{Z}_4$
			$[1,a^2+a+4,-a^2+1,-19a^2+24a-25,-6a^2-78a+39]$	0	$\mathbb{Z}_2\times\mathbb{Z}_2$
			$[1,a^2+a+4,-a^2+1,a^2-11a,128a^2-312a+218]$	0	$\mathbb{Z}_2$
$435b$	1169	$-5a^2+a-8$	$[a,2a^2+a+1,a^2,-a^2+14a-7,-11a^2+26a-16]$	0	$\mathbb{Z}_2$
			$[a,2a^2+a+1,a^2,4a^2+84a+43,-317a^2+211a+298]$	0	$\mathbb{Z}_2$

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