

ON LOCALLY INFINITE CAYLEY GRAPHS OF THE INTEGERS

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ABSTRACT. We compute metric properties of Cayley graphs of the integers with respect to various infinite generating sets. When the generating set S is the set of all powers of a prime, we find explicit formulas for the smallest positive integer of a given length. We also prove that such graphs are infinite dimensional in a strong sense by showing that they fail to have Yu's property A. Finally, we consider more general generating sets and relate geometric properties of these Cayley graphs to deep unsolved problems in number theory.

1. INTRODUCTION

Let G be a group. If we fix a generating set S for G , then we can construct a graph $\Gamma = \Gamma(G, S)$ corresponding to G and S by taking the vertices of Γ to be the elements of G itself and connecting any two vertices g and h by an edge whenever $gs = h$ for some element $s \in S$. We can consider the graph Γ as a metric space by setting the length of each edge to be 1 and taking the path metric on Γ ; this procedure turns the algebraic object G into a geometric object Γ . The drawback of this approach is that different choices of S can lead to wildly different geometric objects; however, when G is a finitely generated group, any two choices of finite generating sets S and S' will give rise to two graphs that are the same on the large scale, see Section 2.

In this paper, we investigate the extent to which different choices of infinite generating sets S can change the graph $\Gamma(G, S)$ when $G = \mathbb{Z}$.

We are chiefly interested in generating sets that are closed under additive inverses and are closed under taking powers. The simplest such generating set is the collection $S_g = \{1, \pm g, \pm g^2, \pm g^3, \dots\}$. We denote $\Gamma(\mathbb{Z}, S_g)$ by C_g . Edges in the graph C_g connect each vertex to infinitely many other vertices, see Figure 1. It is not difficult to see that the graphs C_2 and C_3 are distinct, but the question of whether they are the same on the large scale remains open [9] and motivates our study of the metric properties of these graphs.

When g is prime, the study of these graphs leads to an interesting interplay between geometric group theory (the geometry of the graph) and number theory (the structure of the integer vertices). For example, in C_2 we can ask for the value of the smallest $n > 0$ (in the usual ordering of the integers) at distance d from 0. Looking at Figure 1, one can see that 3 is the smallest positive number that is at distance 2 from 0; extrapolating this figure further, one can verify that 11 is the smallest positive integer at distance 3 from 0. This investigation for all bases $g > 1$ gives rise to the formulas for $\lambda_g(d)$ in Section 3.

These formulas for $\lambda_g(d)$ show that these graphs all have infinite diameter. It is natural to wonder how large these graphs are in some precise sense. It was shown that the large-scale dimension of these graphs is infinite [1]. We show that these graphs have interesting

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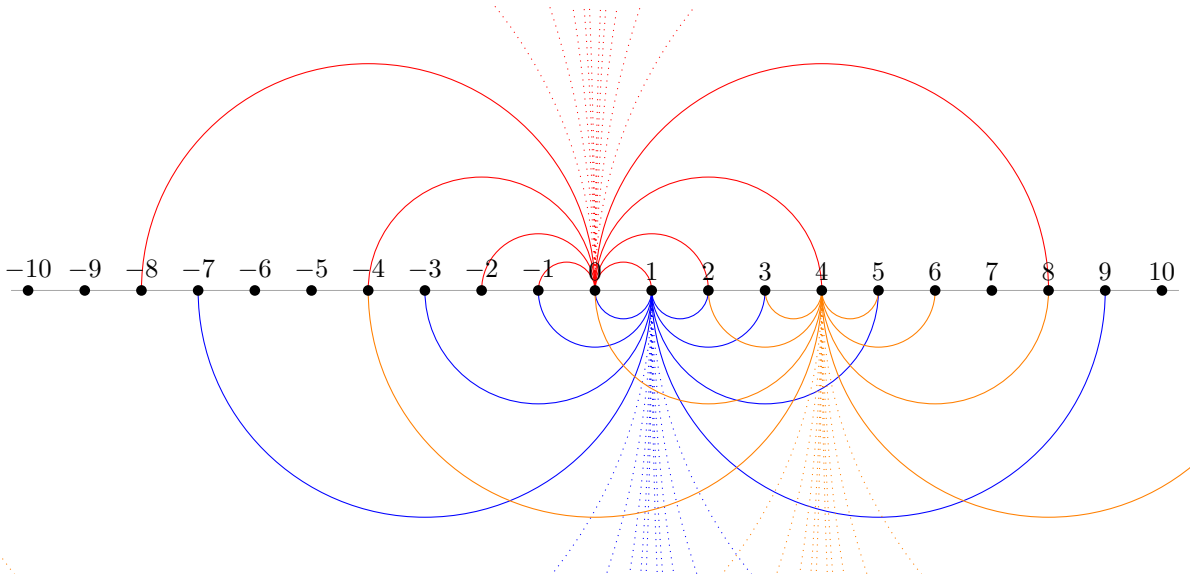


FIGURE 1. Some edges emanating from 0 in red, from 1 in blue, and from 4 in orange in the graph C_2 . Each vertex is incident with infinitely many edges. Observe that the distance from 0 to 3 is 2.

structures we call k -prisms. Any graph with k -prisms has infinite dimension in a strong sense. This allows us to generalize that result and show that these graphs fail to have Yu's property A.

In the final section of the paper, we investigate more general generating sets for collections P of positive integers. These generating sets take the form $S_P = \bigcup_{g \in P} S_g$. For such generating sets, questions about the diameter of the resulting graphs are already interesting and difficult. In particular, we show that when P is the set of all primes, the diameter of the resulting graph is either 3 or 4; moreover if Goldbach's conjecture holds, it is 3. We also conduct numerical investigations to narrow the search for the smallest positive length-3 integer in the graph $\Gamma(\mathbb{Z}, S_P)$ with P equal to the set of all primes.

2. PRELIMINARIES

Let G be a group with a fixed generating set S . We will assume that the identity is not in S and that S is symmetric in the sense that $s \in S$ implies $s^{-1} \in S$. We define a graph $\Gamma = \Gamma(G, S)$, called the *Cayley graph of G with respect to S* as in the introduction: the vertices of Γ are in one-to-one correspondence with the elements of G . We connect the elements g and h in G with an edge precisely when there is an $s \in S$ such that $gs = h$. We can view G as a metric space by taking the edge-length metric d_S on Γ . More precisely, if g and $h \in G$, then write $g^{-1}h$ as a word in the elements of S with minimal length, say $g^{-1}h = s_1 s_2 \cdots s_n$. Then $g s_1 s_2 \cdots s_n = h$, and there is a path of length n between g and h in the Cayley graph. Thus $d_S(g, h) = n$.

Alternatively, we could define a norm $\|\cdot\|_S$ on G with respect to the generating set S by setting $\|g\|_S = \min\{n: s_1 s_2 \cdots s_n = g, s_i \in S\}$. The distance $d_S(g, h) = \|g^{-1}h\|_S$ defines a metric on G called the *left-invariant word metric*.

Example 2.1. The dihedral group of order 10 can be described as the symmetries of a regular pentagon. It can be presented in terms of generators and relations as

$$D_{2,5} = \langle r, s \mid r^5 = 1, s^2 = 1, rs = sr^{-1} \rangle.$$

With $S = \{r, r^{-1}, s\}$, we compute $d_S(rs, r^2sr^{-1}) = 2$. Indeed, using the relations of $D_{2,5}$, we find

$$d_S(rs, r^2sr^{-1}) = \|s^{-1}r^{-1}r^2sr^{-1}\|_S = \|s^{-1}rsr^{-1}\|_S = \|s^{-1}sr^{-1}r^{-1}\|_S = \|r^{-2}\|_S = 2.$$

The Cayley graph of the dihedral group of order 10 with $S = \{s, r, r^{-1}\}$ is indicated on the left-hand side of Figure 2. Notice that vertices are connected by (undirected) edges precisely when the two vertices differ by right multiplication by an element of $\{s, r, r^{-1}\}$. In the right-hand side of that figure, we indicate a geodesic between the elements $rs = sr^4$ and $r^2sr^{-1} = sr^2$.

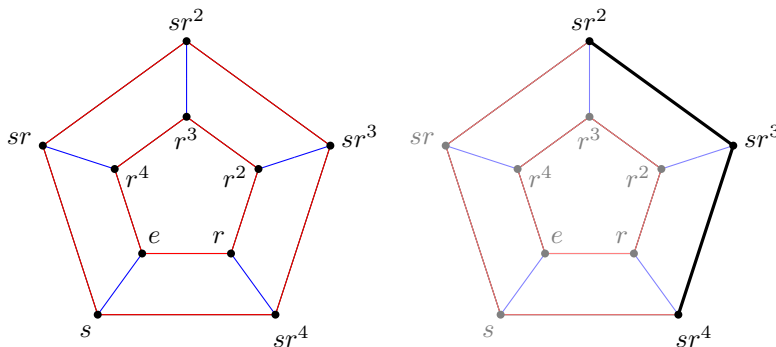


FIGURE 2. On the left the Cayley graph of the dihedral group of order 10 with the generating set $\{s, r, r^{-1}\}$. Left-multiplication by s is shown by a red edge, while left-multiplication by r or r^{-1} is shown in blue. On the right is a visualization of the distance between sr^4 and sr^2 .

While it is true that different choices of generating sets can give rise to vastly different metric spaces, in the case that G is finitely generated, any two finite generating sets give rise to metric spaces that are quasi-isometric (see below). Quasi-isometry is one of several notions of large-scale equivalence of metric spaces, but it is the only one that we will consider.

Definition 2.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \rightarrow Y$ is said to be a *quasi-isometric embedding* if there exist constants $A \geq 1$ and $B \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{A}d_Y(f(x), f(x')) - B \leq d_X(x, x') \leq Ad_Y(f(x), f(x')) + B.$$

Definition 2.3. The metric spaces (X, d_X) and (Y, d_Y) are said to be *quasi-isometric* if there is a quasi-isometric embedding $f: X \rightarrow Y$, and there is some $K \geq 0$ such that for every $y \in Y$ there is some $x \in X$ so that $d_Y(y, f(x)) \leq K$. In this case, we describe the map f as a *quasi-isometry*.

Example 2.4. The Cayley graph $\Gamma(G, S)$ is quasi-isometric to the group G in the metric d_S described above via the identity map, with $A = 1$, $B = 0$, and $K = 0$.

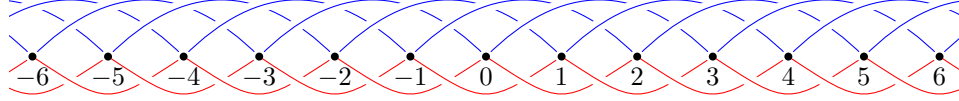


FIGURE 3. The Cayley graph $\Gamma(\mathbb{Z}, \{\pm 2, \pm 3\})$ is shown. Notice that the identity map from \mathbb{Z} in the standard metric to this Cayley graph is a surjective quasi-isometry with $A = 2$, $B = 0$, and $K = 0$.

Example 2.5. Let G be a finitely generated group with finite (symmetric) generating sets S and T . The identity map $\text{id}: G \rightarrow G$ is a quasi-isometric embedding from (G, d_S) to (G, d_T) with A equal to the maximum of the of the longest S -path that represents an element of T and the longest T -path that represents an element of S , and $B = 0$. Moreover, $\text{id}(G) = G$, so with $K = 0$ we see that the identity map is a quasi-isometry.

3. METRIC PROPERTIES OF C_g

We are interested in the case of infinite generating sets for $G = \mathbb{Z}$. In particular, let $g > 0$ be an integer, and denote by $C_g = \Gamma(\mathbb{Z}, S_g)$ the Cayley graph of \mathbb{Z} with the generating set $S_g = \{\pm g^i: i \in \mathbb{Z}_{\geq 0}\}$. Let $d_g = d_{S_g}$ denote the corresponding edge-length metric. We denote the distance $d_g(0, n)$ by $\ell_g(n)$ and refer to this as the *length* of n .

The following theorems of Nathanson [9] give a method of computing length in C_g .

Theorem 3.1 ([9, Theorem 3]). *Let g be an even positive integer. Every integer n has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

such that

- (1) $\varepsilon_i \in \{0, \pm 1, \pm 2, \dots, \pm g/2\}$ for all nonnegative integers i ,
- (2) $\varepsilon_i \neq 0$ for only finitely many nonnegative integers i ,
- (3) if $|\varepsilon_i| = g/2$, then $|\varepsilon_{i+1}| < g/2$ and $\varepsilon_i \varepsilon_{i+1} \geq 0$.

Moreover, n has length

$$\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|.$$

Theorem 3.2 ([9, Theorem 6]). *Let g be an odd integer, $g \geq 3$. Every integer n has a unique representation in the form*

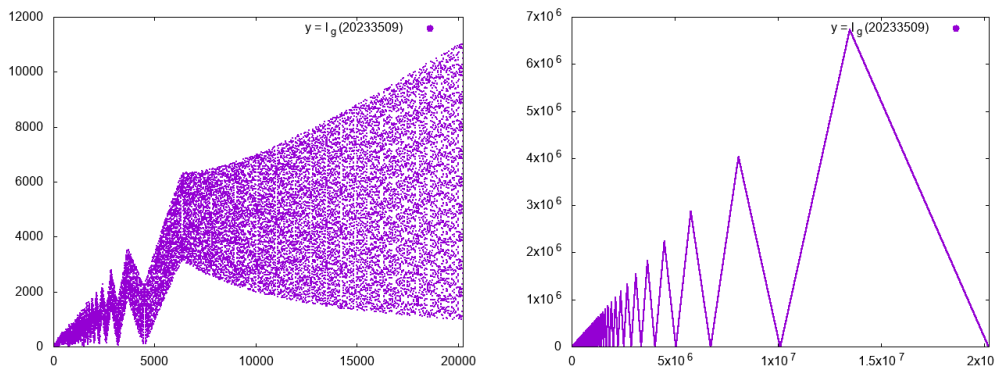
$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

such that

- (1) $\varepsilon_i \in \{0, \pm 1, \pm 2, \dots, \pm (g-1)/2\}$ for all nonnegative integers i ,
- (2) $\varepsilon_i \neq 0$ for only finitely many nonnegative integers i .

Moreover, n has length

$$\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|.$$


 FIGURE 4. Plots of $y = \ell_g(20233509)$ as a function of g .

For any integers n and $g > 2$, Theorems 3.1 and 3.2 give a unique g -adic expression for n that realizes a geodesic path from 0 to n . Thus there is $N > 0$ such that $n = \sum_{i=0}^N \varepsilon_i g^i$, $\varepsilon_N \neq 0$, and $\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$. We call $n = \sum_{i=0}^N \varepsilon_i g^i$ the *minimal g -adic expansion*, and denote it by

$$[n]_g = [\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N].$$

It is interesting to look at how $\ell_g(n)$ varies as a function of g . See Figure 4. We chose a random number $n = 20,233,509$, and produced a plot of $y = \ell_g(n)$ for a range of values for g . For g sufficiently large, we have $\ell_g(n) = n$, but it appears that interesting things happen along the way.

Example 3.3. The minimal 5-adic expansion of 46 is $[46]_5 = [1, -1, 2]$, so $46 = 1 - 5 + 2 \cdot 5^2$, and $\ell_5(46) = 1 + 1 + 2 = 4$.

We denote by $\lambda_g(h)$ the smallest positive integer of length h in C_g . We find an explicit formula for λ_g in Theorems 3.4 and 3.5 below using Nathanson's g -adic representation [9] of positive integers. The first few values are tabulated in Table 1. We remark that the values of λ_2 show up in The On-Line Encyclopedia of Integer Sequences (OEIS) as A007583, and the values of λ_3 show up as A007051. The sequences of values for λ_p for other primes p do not seem to appear. Figure 5 shows the integers less than 10,000 and their 19-length, together with the graph of $y = \lambda_{19}(x)$.

Theorem 3.4. Let $g > 0$ be an odd integer, and let $k > 0$ be an integer. Let $q = \lfloor \frac{2k}{g-1} \rfloor$, and let $r = k \bmod \frac{g-1}{2}$ so that $k = q(\frac{g-1}{2}) + r$. Then

$$\lambda_g(k) = \begin{cases} \frac{1 + g^{q-1}(g-2)}{2} & \text{if } r = 0, \\ \frac{1 + g^q(2r-1)}{2} & \text{otherwise.} \end{cases}$$

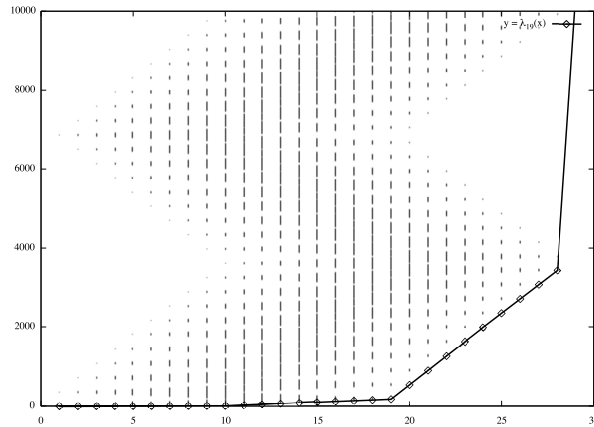
Proof. We consider each case separately.

Case 1. ($r = 0$): Let $q = \lfloor \frac{2k}{g-1} \rfloor$ and $z = \frac{1+g^{q-1}(g-2)}{2}$. First, we show $\ell_g(z) = k$. A straightforward computation shows that the minimal g -adic expansion of z is given by

$$[z]_g = \left[\frac{1 + g^{q-1}(g-2)}{2} \right]_g = [-b, -b, \dots, -b, b],$$

TABLE 1. First few values of $\lambda_p(k)$ for primes $p < 30$.

k	2	3	5	7	11	13	17	19	23	29
1	1	1	1	1	1	1	1	1	1	1
2	3	2	2	2	2	2	2	2	2	2
3	11	5	3	3	3	3	3	3	3	3
4	43	14	8	4	4	4	4	4	4	4
5	171	41	13	11	5	5	5	5	5	5
6	683	122	38	18	6	6	6	6	6	6
7	2731	365	63	25	17	7	7	7	7	7
8	10923	1094	188	74	28	20	8	8	8	8
9	43691	3281	313	123	39	33	9	9	9	9
10	174763	9842	938	172	50	46	26	10	10	10
11	699051	29525	1563	515	61	59	43	29	11	11
12	2796203	88574	4688	858	182	72	60	48	12	12
13	11184811	265721	7813	1201	303	85	77	67	35	13
14	44739243	797162	23438	3602	424	254	94	86	58	14
15	178956971	2391485	39063	6003	545	423	111	105	81	15
16	715827883	7174454	117188	8404	666	592	128	124	104	44
17	2863311531	21523361	195313	25211	1997	761	145	143	127	73
18	11453246123	64570082	585938	42018	3328	930	434	162	150	102
19	45812984491	193710245	976563	58825	4659	1099	723	181	173	131
20	183251937963	581130734	2929688	176474	5990	3296	1012	542	196	160

FIGURE 5. The integers up to 10,000 whose 19-lengths are x are shown together with the graph of $y = \lambda_{19}(x)$.

where $b = \frac{g-1}{2}$, and there are q digits in the expansion $[z]_g$. Note that in this case since $r = 0$, we have $q = \lfloor \frac{2k}{g-1} \rfloor = \frac{2k}{g-1}$. It follows that $\ell_g(z) = b \cdot q = k$, as desired.

Suppose $x \leq z$ is a positive integer of length k . Let $[x]_g = [\varepsilon_0, \dots, \varepsilon_n]$ be the minimal g -adic expansion of x . By Theorem 3.2, we have $|\varepsilon_i| \leq b$ for all i . There must be q digits in the expansion $[x]_g$ so that $n \leq q - 1$, since if $n > q - 1$ and $\varepsilon_n > 0$,

then $x > z$. Since $\ell_g(x) = \ell_g(z) = k$, we must have

$$\ell_g(x) - \ell_g(z) = \sum_{i=0}^{q-1} (b - |\varepsilon_i|) = 0.$$

But this means $n = q - 1$ and $|\varepsilon_i| = b$ for all i . Since $x \leq z$, we cannot have a number less than $-\frac{g-1}{2}$ for ε_i , $i = 0, \dots, q - 2$, thus we must have $-\frac{g-1}{2} = \varepsilon_i$. Similarly, we cannot have a number larger than $\frac{g-1}{2}$ for ε_{q-1} . We cannot have $\varepsilon_{q-1} < \frac{g-1}{2}$ without changing the length. Therefore $z = x$.

Case 2: ($r \neq 0$): With $q = \lfloor \frac{2k}{g-1} \rfloor$ and $z = \frac{1+g^q(2r-1)}{2}$ we see

$$[z]_g = \left[\frac{1 + g^q(2r - 1)}{2} \right]_g = [-b, -b, \dots, -b, r],$$

where there are $q + 1$ digits in the minimal g -adic expansion. It is clear that $\ell_g(z) = k$. Suppose $x \leq z$ is a positive integer of length k . Theorem 3.2 tells us $[x]_g = [\varepsilon_0, \dots, \varepsilon_n]$ with $|\varepsilon_i| \leq b$ for all i . It is helpful to note that $z < [-b, -b, \dots, -b, b] < [-b, -b, \dots, -b, 1]$ where there are $q + 1$ digits in the middle expression and $q + 2$ digits on the right. Hence if $n > q$ and $\varepsilon_n > 0$ then $x > z$. Thus since $\ell_g(x) = \ell_g(z)$ we have

$$r - \varepsilon_q + \sum_{i=0}^{q-1} \frac{g-1}{2} - |\varepsilon_i| = 0.$$

This means each of the summands must be 0. Hence $\varepsilon_q = r$ and $\varepsilon_i = -\frac{g-1}{2}$ for all i . Therefore, z is truly the smallest integer of length k . \square

Theorem 3.5. *Let $g > 0$ be an even integer, and let $k > 0$ be an integer. Let $r = k \bmod g - 1$. Define integers q , A , and B by*

$$q = \begin{cases} \lfloor \frac{k}{g-1} \rfloor - 1 & \text{if } r = 0, \\ \lfloor \frac{k}{g-1} \rfloor & \text{if } 1 \leq r \leq g - 1; \end{cases}$$

$$A = \begin{cases} g/2 & \text{if } r = 0 \text{ or } r > g/2, \\ r & \text{otherwise;} \end{cases}$$

$$B = \begin{cases} g/2 - 1 & \text{if } r = 0, \\ r - g/2 & \text{if } r > g/2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\lambda_g(k) = \frac{g(1 - g^{2q})}{2(1 + g)} + Ag^{2q} + Bg^{2q+1}.$$

Proof. Let $b = g/2$. One can see that

$$z := \frac{g(1 - g^{2q})}{2(1 + g)} + Ag^{2q} + Bg^{2q+1} = [-b, -(b-1), -b, -(b-1), \dots, A, B]_g,$$

where there are $2q + 2$ digits in the representation. Suppose first $B \neq 0$ and $x \leq z$ has length k . By Theorem 3.1, $[x]_g = [\varepsilon_0, \dots, \varepsilon_n]$ where $\varepsilon_i \in \{0, \pm 1, \pm 2, \dots, \pm g/2\}$ for all $0 \leq i \leq n$ and if $|\varepsilon_i| = g/2$, then $|\varepsilon_{i+1}| < g/2$ and $\varepsilon_i \varepsilon_{i+1} \geq 0$. Notice $z \leq [0, 0, \dots, 1]_g$ where there are $2q + 3$ digits in the representation on the right. Hence, it is clear that $n \leq 2q + 2$. \square

4. C_g FAILS TO HAVE PROPERTY A

Richard E. Schwartz [9, Problem 6] asked whether (C_2, d_2) and (C_3, d_3) are quasi-isometric. This question could be answered in the negative by finding a quasi-isometry invariant that distinguishes them. Therefore the question of determining quasi-isometry invariants for C_g for different values of g is an interesting one. Adams, Gulbrandsen, and Vasilevska show that the asymptotic dimension of C_g is infinite for $g \in \{2, 3\}$ [1, Theorem 5]. The asymptotic dimension is a large-scale analog of dimension that belongs to a family conditions on covers of metric spaces. For metric spaces with infinite asymptotic dimension, it is interesting to consider another quasi-isometry invariant: Yu's property A.

Definition 4.1 ([14]). A (discrete) metric space X is said to have *property A* if for all $R > 0$ and all $\varepsilon > 0$, there exists a family $\{A_x\}_{x \in X}$ of finite, non-empty subsets of $X \times \mathbb{Z}_{\geq 1}$ such that

- (1) for all $x, y \in X$ with $d(x, y) \leq R$, we have $\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \leq \varepsilon$, and
- (2) there exists a $B > 0$ such that for every $x \in X$, if $(y, n) \in A_x$, then $d(x, y) \leq B$.

Here $\#A$ is the cardinality of A and $A_x \Delta A_y$ denotes the symmetric difference.

We use an example of Nowak [10] to show that C_g fails to have property A for every integer $g > 1$. This can be interpreted as saying that C_g is infinite dimensional in a strong sense.

Example 4.2. We note that any tree has property A. To see this we need to show that for every $\varepsilon > 0$ and for all $R > 0$ there exists a family $\{A_x\}_{x \in T}$ of finite subsets of $T \times \mathbb{N}$ and an $r > 0$ such that

- (1) $\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} < \varepsilon$, if $d(x, y) \leq R$, and
- (2) $A_x \subset B(x, r) \times \mathbb{N}$ for every $x \in T$.

To show this, we first fix a geodesic ray γ_0 and suppose that for all $x \in T$, γ_x is a ray that begins at x with the property that $\gamma_0 \cap \gamma_x$ is a geodesic ray. It is easy to see that such a γ_x exists and is unique in a tree. Now let $\varepsilon > 0$ and $R > 0$ be given and take $r \geq \frac{R}{\varepsilon}$. Put $A_x = \gamma_x([0, r]) \times \{1\}$. Then, since $(A_x \Delta A_y) = (A_x \setminus A_y) \cup (A_y \setminus A_x)$ and both A_x and A_y intersect γ_0 except at possibly R places, we have that $\#(A_x \Delta A_y) < 2R$. Furthermore, $\#(A_x \cap A_y) \geq \frac{2R}{\varepsilon}$, and so $\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \leq \varepsilon$. Finally, it is clear that $A_x \subset B(x, r) \times \mathbb{N}$.

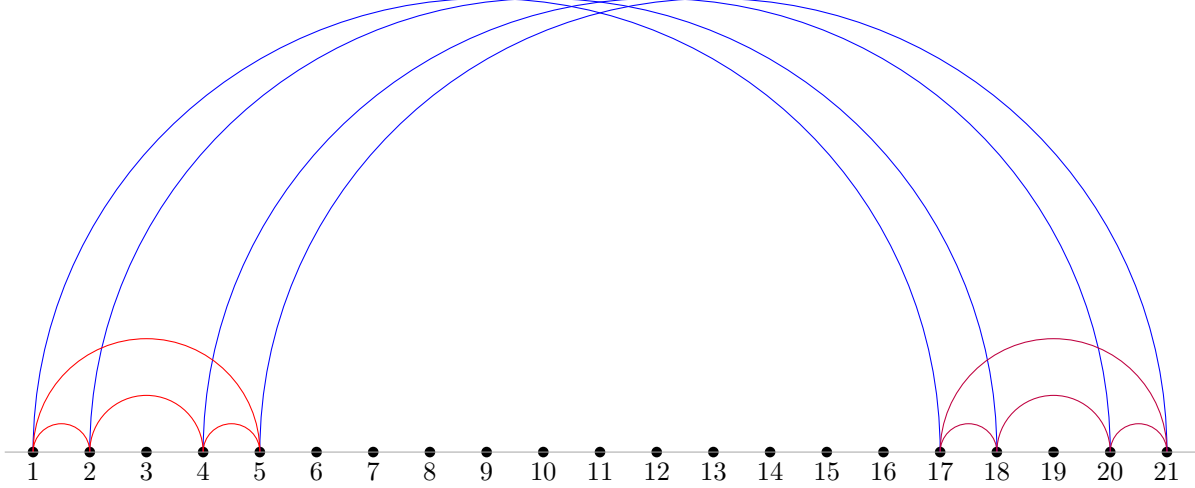
Example 4.3. Let $\{0, k\}^n$ be the set of vertices of an n -dimensional cube at scale k endowed with the ℓ_1 -metric. The disjoint union $\coprod_{n=1}^{\infty} \{0, k\}^n$ can be metrized in such a way that it is a locally finite metric space that fails to have property A [10].

In order to utilize Example 4.3 we define the notion of k -prisms. We show that a metric space with k -prisms has disjoint isometric copies of $\{0, k\}^n$ for all n . Thus, the existence of k -prisms is an obstruction to having property A.

Definition 4.4. Let k be a positive integer. We say that a metric space X has k -prisms if for any finite set $F \subset X$ there exists an isometry T such that

- (1) $T(F) \cap F = \emptyset$ and
- (2) $d(x, T(y)) = k + d(x, y)$ for all $x, y \in F$.

Remark 4.5. We note that if a metric space X has k -prisms then X has nk -prisms for all $n \in \mathbb{N}$. To show this we use induction on n . The base case is trivial since X having k -prisms means X has $1k$ -prisms. Now we assume the X has $(n-1)k$ -prisms and we wish to show


 FIGURE 6. Constructing a 1-prism over $\{1, 2, 4, 5\}$ in C_2 .

that X has nk -prisms. Then since X has $(n-1)k$ -prisms for any finite subset F , there is an isometry T_{n-1} so that $T_{n-1}(F) \cap F = \emptyset$ and $d(x, T_{n-1}(y)) = (n-1)k + d(x, y)$. Since X has k -prisms, we can find an isometry T_n taking the set $F \cup T_{n-1}(F)$ to an isometric copy so that each vertex of $F \cup T_{n-1}(F)$ is at distance k from its image. Thus, if we restrict T_n to the image $T_{n-1}(F)$, we see that $d(x, T_n(T_{n-1}(y))) = k + d(x, T_{n-1}(y)) = k + (n-1)k + d(x, y) = nk + d(x, y)$ and $T_n(F) \cap F = \emptyset$. Therefore X has nk -prisms for all $n \geq 1$.

Lemma 4.6. *The space C_g has k -prisms for every k .*

Proof. Let $F \neq \emptyset$ be an arbitrary finite subset of C_g . By Remark 4.5, it suffices to find an isometry T such that $F \cap T(F) = \emptyset$ and $d(x, T(y)) = 1 + d(x, y)$ for all $x, y \in F$. Now we know from Theorem 3.1 and Theorem 3.2 that for any g we have a unique representation for an integer x of the form,

$$x = \sum_{i=0}^{\infty} \varepsilon_i g^i,$$

where the requirements of the ε_i change depending on whether g is even or odd, and $\varepsilon_i = 0$ for all but finitely many indices. Since F is finite, there is some positive integer m so that $\varepsilon_i = 0$ for all $i > m$ for each $x \in F$. Now, we define an isometry T that takes $x = \sum_{i=0}^m \varepsilon_i g^i$ to $T(x) = \sum_{i=0}^m \varepsilon_i g^i + g^{m+2}$. We note that choosing g^{m+1} is not sufficient since we cannot expect this expression to be in the canonical form. Clearly T is an isometry. Now we see that

$$d(x, T(y)) = d\left(\sum_{i=0}^m \varepsilon_i g^i, \sum_{i=0}^m \delta_i g^i + g^{m+2}\right) = \sum_{i=0}^m |\delta_i - \varepsilon_i| + 1 = d(x, y) + 1.$$

So $d(x, T(y)) = d(x, y) + 1$, and by construction $F \cap T(F) = \emptyset$. Therefore C_g has k -prisms for each k . \square

In Figure 4 we give an example of a 1-prism constructed by the method of this proof over a set in C_2 .

Lemma 4.7. *The space C_g contains an infinite geodesic ray.*

Proof. We note that since C_g has 1-prisms we can start with a single point $x \in C_g$. We use the T_n from Remark 4.5 and obtain an infinite geodesic ray made up of the points $\{x, T_1(x), T_2(T_1(x)), T_3(T_2(T_1(x))), \dots\}$. This sequence has the property that $d(x, T_n(x)) = n + d(x, T_1(x))$ and $d(T_i(x), T_j(x)) = i - j$. Thus we have an infinite geodesic ray. \square

Lemma 4.8. *If $X \neq \emptyset$ has k -prisms for some k , then X contains an isometric copy of $\{0, k\}^n$. In particular, the space C_g contains an isometric copy of $\{0, k\}^n$ for each n and for each k .*

Proof. Let X be a metric space with k -prisms for some k . In order to show that X has an isometric copy of $\{0, k\}^n$, we use induction. To this end we note that if X has k -prisms then for any finite subset $F \subset X$, there exists an isometry T such that $T(F) \cap F = \emptyset$ and $d(x, T(y)) = k + d(x, y)$ for all $x, y \in F$.

Taking any $a \in X$, we apply the k -prism condition to find a map T such that $d(a, T(a)) = k$. Clearly, $\{a, T(a)\}$ is an isometric copy of $\{0, k\}$. Now we assume that X has an isometric copy, C , of $\{0, k\}^{n-1}$. Then $T(C)$ is also an isometric copy of $\{0, k\}^{n-1}$. We claim that $C \cup T(C)$ is an isometric copy of $\{0, k\}^n$. This follows from the fact that $d(x, T(y)) = k + d(x, y)$ for all $x, y \in C \cup T(C)$. Therefore if X has k -prisms then X has an isometric copy of $\{0, k\}^n$.

In particular, since C_g has k -prisms for all k , C_g has an isometric copy of $\{0, k\}^n$ for all k . \square

Theorem 4.9. *Let X be any discrete metric space with k -prisms for some $k \geq 1$, then X does not have property A.*

Proof. First we note that if Z has property A and if $Y \subset Z$ then Y has property A. Now given Lemma 4.6 we take Y to be the graph in Example 4.3, which does not have property A. Then since X has k -prisms for some k , we see that X has an infinite geodesic ray and an isometric copy of $\{0, k\}^n$ for each n . Thus we have disjoint k -scale n -cubes for every n and so $Y \subset X$. Since Y does not have property A, X cannot have property A. \square

Corollary 4.10. *Let $g > 1$ be an integer. Then C_g fails to have Yu's property A.*

Definition 4.11. The direct sum of a sequence $\{G_n\}_{n=1}^{\infty}$ of groups G_n is the set of all sequences $\{g_n\}_{n=1}^{\infty}$ where $g_n \in G_n$ and g_n is equal to the identity element of G_n for all but a finite set of indices. This is denoted $\bigoplus_{n=1}^{\infty} G_n$.

Example 4.12. We remark that the fact that k is fixed in the definition of k -prisms is important. We describe a space X with property A that contains an isometric copy of $\{0, n!\}^n$ for each n . This space has the property that for every finite subset $F \subset X$ there is a k and an isometry $T : F \rightarrow X$ such that $d(x, T(y)) = k + d(x, y)$ for all $x, y \in F$, yet X does not have k -prisms for any fixed k . Our example is $X = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ with the metric $d(x, y) = \sum_{i=1}^{\infty} i|x_i - y_i|$, which Yamauchi showed has property A [13]. To show X contains an isometric copy of $\{0, n!\}^n$ for each n , we define an isometry $f : \{0, n!\}^n \rightarrow X$ by $f(t_1, \dots, t_n) = (\frac{t_1}{1}, \frac{t_2}{2}, \dots, \frac{t_n}{n}, 0, 0, \dots)$. Then, since each t_i is either 0 or $n!$, it follows that each t_i is divisible by i , so $\frac{t_i}{i} \in \mathbb{Z}$. Also, for any s and t in $\{0, n!\}^n$,

$$d_{\ell_1}(s, t) = \sum_{i=1}^n |s_i - t_i| = \sum_{i=1}^n i \left| \frac{s_i}{i} - \frac{t_i}{i} \right| = d(f(s), f(t)).$$

Thus X contains an isometric copy of $\{0, n!\}^n$.

5. METRIC PROPERTIES OF C_P

One can consider more general infinite generating sets for \mathbb{Z} . Let P be a set of positive integers. Let $C_P = \Gamma(\mathbb{Z}, S_P)$ denote the Cayley graph of \mathbb{Z} with the generating set

$$S_P = \bigcup_{a \in P} \{\pm a^i : i \in \mathbb{Z}_{\geq 0}\}.$$

We give C_P the edge-length metric d_{S_P} , and use $\ell_P(x)$ to denote the length of x in the metric d_{S_P} ; i.e., $\ell_P(x) = d_{S_P}(0, x)$.

It is natural to consider determining the length function for C_P and whether C_P has property A for various collections P ; however, it is already a difficult problem to compute the diameter of C_P with respect to a generating set with more than one element.

Question 5.1. Let P be a set of primes. Let $\lambda_P(h)$ denote the smallest positive integer of length h in C_P . Compute the function $\lambda_P(h)$.

There are partial results addressing Question 5.1 when $\#P < \infty$. Hadju and Tijdeman [6] prove that $\exp(ck) < \lambda_P(k) < \exp((k \log k)^C)$, with some constant c depending on P and an absolute constant C .

Nathanson [8] gives a class of generating sets for \mathbb{Z} whose arithmetic diameters are infinite.

Theorem 5.2 ([8, Theorem 5]). *If P is a finite set of positive integers, then C_P has infinite diameter.*

Theorem 5.2 does not hold for infinite P . The *ternary Goldbach conjecture* states that every odd integer n greater than 5 can be written as the sum of three primes. H. A. Helfgott's proof [7, Main Theorem] of this implies if \mathcal{P} is the set of all primes, then $C_{\mathcal{P}}$ is at most 4.

Theorem 5.3. *Let \mathcal{P} be the set of all primes. The diameter of $C_{\mathcal{P}}$ is 3 or 4.*

Proof. It is easy to see that $\ell_{\mathcal{P}}(n) = 1$ for $n \in \{1, 2, 3, 4, 5\}$. Helfgott [7, Main Theorem] proves that every odd integer greater than 5 can be written as the sum of three primes. Since every even integer greater than 4 can be expressed as 1 less than an odd integer greater than 5, we have that $\ell_{\mathcal{P}}(n) \leq 4$ for all $n \in \mathbb{Z}$.

Since not every integer is a prime power, the diameter of $C_{\mathcal{P}}$ is at least 2. To show that the diameter is not 2, it suffices to produce an integer that is not a prime power and cannot be expressed as the sum or difference of prime powers, where the prime power $p^0 = 1$ is allowed. Such integers are surprisingly hard to find. First note that the Goldbach conjecture asserts that every even integer greater than 2 can be expressed as the sum of two primes. This has been computationally verified integers less than $4 \cdot 10^{18}$ [11]. It follows that $\ell_{\mathcal{P}}(n) \leq 2$ for even integers $n < 4 \cdot 10^{18}$. Thus a search for an integer of length 3 should be restricted to odd integers. An odd integer M is length 3 if

- (1) M is not prime power;
- (2) $|M \pm 2^n|$ is not prime power for all $n \geq 0$.

Cohen and Selfridge [4, Theorem 2] use covering congruences to prove the existence of an infinite family of integers M satisfying item (2) and give an explicit 94-digit example of such an integer. Zhi-Wei Sun [12] adapts their work to produce a much smaller example. Specifically, let

$$M = 47867742232066880047611079, \quad \text{and let } N = 66483084961588510124010691590.$$

Sun proves that if $x \equiv M \pmod N$, then x is not of the form $|p^a \pm q^b|$ for any primes p, q and nonnegative integers a, b ¹. We use Atkin and Morain's ECPP (Elliptic Curve Primality Proving) method [2] implemented by Morain in MAGMA [3] to look in this congruence class for an element that is provably not a prime power. We find that M and $M + N$ are prime, but

$$\begin{aligned} M + 2N &= 133014037665409087128068994259 \\ &= 23 \cdot 299723 \cdot 19295212676140402555471 \end{aligned}$$

is not a prime power. Thus $\ell_{\mathcal{P}}(M + 2N) = 3$, and the result follows. \square

Remark 5.4. Assuming Goldbach's conjecture, the diameter of $C_{\mathcal{P}}$ is 3.

It is still an open problem to find the smallest integer n that is not of the form $|p^a \pm q^b|$, for any primes p, q and nonnegative integers a, b [5, A19]. Explicit computations [4, 12] show that the smallest such integer must be larger than 2^{25} . Such elements, if not prime powers, would have \mathcal{P} -length 3. We have extended slightly their computation and confirmed that $\ell_{\mathcal{P}}(n) < 3$ for all $n < 58,164,433 \approx 2^{25.79}$. For

$$n = 58164433 = 4889 \cdot 11897,$$

we could not show $\ell_{\mathcal{P}}(n) = 2$. It is possible that this integer is the smallest positive integer of length 3.

Theorem 5.5. *Let \mathcal{P} denote the set of all primes, and let $S \subset \mathcal{P}$ be a finite subset. Let $\mathcal{P}' = \mathcal{P} \setminus S$. If $R = \max_{p \in S} \{\ell_{\mathcal{P}'}(p)\}$, then $\text{diam}(C_{\mathcal{P}'}) \leq 3 \max\{R, 3\} + 1$. In particular, the diameter of $C_{\mathcal{P}'}$ is finite.*

Proof. First note that if p is a prime in S , then $\ell_{\mathcal{P}'}(p) \leq R$. If p is a prime not in S , then by the ternary Goldbach conjecture [7, Main Theorem] we have $\ell_{\mathcal{P}'}(p) \leq 3$. Thus $\ell_{\mathcal{P}'}(p) \leq \max\{R, 3\}$ for any prime p .

Since every even integer is one less than an odd integer, it suffices to show that $\ell_{\mathcal{P}'}(n) \leq 3 \max\{R, 3\}$ for every positive odd integer n . By the ternary Goldbach conjecture, every odd integer $n > 5$ can be expressed as the sum of three primes. Let $n = p + q + r > 5$ be an odd integer for some primes p, q, r . Then

$$\ell_{\mathcal{P}'}(n) \leq \ell_{\mathcal{P}'}(p) + \ell_{\mathcal{P}'}(q) + \ell_{\mathcal{P}'}(r) \leq \max\{R, 3\} + \max\{R, 3\} + \max\{R, 3\},$$

and the result follows. \square

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¹The modulus N given by Sun [12] is incorrectly written as 66483034025018711639862527490.

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