

# ON THE TOPOLOGICAL COMPUTATION OF $K_4$ OF THE GAUSSIAN AND EISENSTEIN INTEGERS

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**ABSTRACT.** In this paper we use topological tools to investigate the structure of the algebraic  $K$ -groups  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$ , where  $i := \sqrt{-1}$  and  $\rho := (1 + \sqrt{-3})/2$ . We exploit the close connection between homology groups of  $\mathrm{GL}_n(R)$  for  $n \leq 5$  and those of related classifying spaces, then compute the former using Voronoi's reduction theory of positive definite quadratic and Hermitian forms to produce a very large finite cell complex on which  $\mathrm{GL}_n(R)$  acts. Our main result is that  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$  have no  $p$ -torsion for  $p \geq 5$ .

## 1. INTRODUCTION

**1.1. Statement of results.** Let  $R$  be the ring of integers of a number field  $F$ . Only very few cases are known where the algebraic  $K$ -group  $K_4(R)$  has been explicitly computed, the first such  $K_4(\mathbb{Z})$  having been determined as recently as 2000 by Rognes [17], building on work of Soulé [18]. The goal of this paper is the explicit topological computation of the torsion (away from 2 and 3) in the groups  $K_4(R)$  for  $R$  one of two special imaginary quadratic examples: the *Gaussian integers*  $\mathbb{Z}[i]$  and the *Eisenstein integers*  $\mathbb{Z}[\rho]$ , where  $i := \sqrt{-1}$  and  $\rho := (1 + \sqrt{-3})/2$ . Our work is in the spirit of Lee–Szczarba [12–14], Soulé [19], and Elbaz-Vincent–Gangl–Soulé [7, 8] who treated  $K_N(\mathbb{Z})$  for small  $N$ , and Staffeldt [20] who investigated  $K_3(\mathbb{Z}[i])$ . As in these works, the first step is to compute the cohomology of  $\mathrm{GL}_n(R)$  for  $n \leq N + 1$ ; information from this computation is then assembled into information about the  $K$ -groups following the program in §1.2. Using these computations we show the following (Theorem 4.1):

**Theorem.** *The orders of the groups  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$  are not divisible by any primes  $p \geq 5$ .*

We remark that this result is not new; in fact, Kolster's work [11] implies the stronger result that  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$  vanish. Indeed, if  $R$  is the ring of integers of a  $CM$  field, then Kolster proved that, assuming the Quillen–Lichtenbaum conjecture, the orders of the groups  $K_{4n}(R)$ ,  $n = 1, 2, 3, \dots$ , can be computed in terms of special values of certain  $L$ -functions. This deep connection between  $K$ -groups and special values of  $L$ -functions is now a theorem, thanks to the celebrated work by Voevodsky [21] and Rost, as put into context in [9].

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2010 *Mathematics Subject Classification.* Primary 19D50; Secondary 11F75.

*Key words and phrases.* Cohomology of arithmetic groups, Voronoi reduction theory, linear groups over imaginary quadratic fields,  $K$ -theory of number rings.

MDS was partially supported by the Humboldt Foundation. PG was partially supported by the NSF under contract DMS 1101640 and DMS 1501832. The authors thank the American Institute of Mathematics where this research was initiated.

Our work, on the other hand, treats  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$  by completely different methods. We only use the definition of the  $K$ -groups and explicit results about the cohomology of the relevant arithmetic groups [6], together with Arletaz's bounds on the kernel of the Hurewicz homomorphism [1], to prove Theorem 4.1. This also explains why our calculations do not allow us to say anything for the primes 2 and 3: both the results of [6] and the injectivity of the Hurewicz map in our cases only hold away from these primes.

**1.2. Outline of method.** In the rest of this introduction we outline the main steps of our argument. These follow the classical approach for computing algebraic  $K$ -groups of number rings due to Quillen [15], which shifts the focus to computing the homology (with nontrivial coefficients) of certain arithmetic groups.

- (i) **(Definition)** By definition the algebraic  $K$ -group  $K_N(R)$  of a ring  $R$  is a particular homotopy group of a topological space associated to  $R$ : we have  $K_N(R) = \pi_{N+1}(BQ(R))$ , where  $BQ(R)$  is a certain classifying space attached to the infinite general linear group  $GL(R)$ . In particular  $BQ(R)$  is the classifying space of the category  $Q(R)$  of finitely generated  $R$ -modules. This is known as Quillen's  $Q$ -construction of algebraic  $K$ -theory [16].
- (ii) **(Homotopy to Homology)** The Hurewicz homomorphism  $\pi_{N+1}(BQ(R)) \rightarrow H_{N+1}(BQ(R))$  allows one to replace the homotopy group by a homology group without losing too much information; more precisely, what may get lost is information about small torsion primes appearing in its finite kernel.
- (iii) **(Stability)** By a stability result of Quillen [15, p. 198] one can pass from  $Q(R)$  to the category  $Q_{M+1}(R)$  of finitely generated  $R$ -modules of rank  $\leq M+1$  for sufficiently large  $M$ . This amounts to passing from  $GL(R)$  to the finite-dimensional general linear group  $GL_{M+1}(R)$ . In the cases at hand, a result of Lee and Szczarba allows to reduce to the case  $M = N$ .
- (iv) **(Sandwiching)** The homology groups to be determined are then  $H_*(BQ_n(R))$  for  $n \leq N+1$ . Rather than computing these directly, one uses the fact that they can be sandwiched between homology groups of  $GL_n(R)$ , where the homology is taken with (nontrivial) coefficients in the Steinberg module  $St_n$  associated to  $GL_n(R)$ .
- (v) **(Equivariant homology)** It has been shown for certain number rings  $R$  that the homology groups  $H_m(GL_n(R), St_n)$  are isomorphic to the equivariant  $GL_n(R)$ -homology of an associated pair (denoted  $(X_n^*, \partial X_n^*)$  in §1.3 below). The standard method to compute the latter uses *Voronoi complexes*. These are relative chain complexes of certain explicit polyhedral reduction domains of a space of positive definite quadratic or Hermitian forms of a given rank, depending respectively on whether  $R = \mathbb{Z}$  or  $R$  is imaginary quadratic.
- (vi) **(Vanishing Results)** There are various techniques to show vanishing of homology groups. As a starting point one has vanishing results for  $H_n(BQ_1)$  as in Theorem 3.1 below, and for  $H_0(GL_n, St_n)$  as in Lee–Szczarba [13], Cor. to Thm 4.1.

For a given  $N$ , using (ii) and knowing the results of (iv)–(vi) for all  $0 \leq n \leq N+1$  is often enough to give a bound  $p \leq B$  on the primes  $p$  dividing the order of the torsion subgroup  $K_{N,\text{tors}}(R)$  of  $K_N(R)$ .

**1.3. Outline of paper.** In this paper the sections work backwards through the method outlined in §1.2 to determine the structure of  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$ . In §2, we describe the computation of the equivariant homology in question and relate it to the Steinberg homology. In §3 we use the results on Steinberg homology and some vanishing results to determine the groups  $H_m(BQ_n(R))$  (i.e., step (iv) above). A key role here is played by Quillen’s stability result (iii) for  $BQ_n$ , as refined by Lee-Szczarba in [13], which serves as a stopping criterion. Finally, in §4 we work out the potential primes entering the kernel of the Hurewicz homomorphism (i.e., step (ii) above), which gives Theorem 4.1.

## 2. HOMOLOGY OF VORONOI COMPLEXES

We first collect the results from [6] concerning the Voronoi complexes attached to  $\Gamma = \mathrm{GL}_n(\mathbb{Z}[i])$  or  $\Gamma = \mathrm{GL}_n(\mathbb{Z}[\rho])$ ; this is the necessary information needed for step (v) from §1.2 above. More details about these computations, including background about how the computations are performed, can be found in [6].

Let  $F$  be an imaginary quadratic field with ring of integers  $R$ , and let  $X_n := \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$  be the symmetric space of  $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ . The space  $X_n$  can be realized as the quotient of the cone of rank  $n$  positive definite Hermitian matrices  $C_n$  modulo homotheties (i.e. non-zero scalar multiplication), and a partial Satake compactification  $X_n^*$  of  $X_n$  is given by adjoining boundary components to  $X_n$  given by the cones of positive semi-definite Hermitian forms with an  $F$ -rational nullspace (again taken up to homotheties). We let  $\partial X_n^* := X_n^* \setminus X_n$  denote the *boundary* of  $X_n^*$ . Then  $\Gamma := \mathrm{GL}_n(R)$  acts by left multiplication on both  $X_n$  and  $X_n^*$ , and the quotient  $\Gamma \backslash X_n^*$  is a compact Hausdorff space.

A generalization—due to Ash [2, Chapter II] and Koecher [10]—of the polyhedral reduction theory of Voronoi [22] yields a  $\Gamma$ -equivariant explicit decomposition of  $X_n^*$  into (Voronoi) cells. Moreover, there are only finitely many cells modulo  $\Gamma$  and we have the following result.

**Proposition 2.1** ([6, Proposition 3.6]). *For  $\Gamma \in \{\mathrm{GL}_n(\mathbb{Z}[i]), \mathrm{GL}_n(\mathbb{Z}[\rho])\}$  and  $m \in \mathbb{Z}$  we have  $H_m^\Gamma(X_n^*, \partial X_n^*, \mathbb{Z}) \simeq H_{m-n+1}(\Gamma, St_n)$ .*

Let  $\Sigma_d^* := \Sigma_d(\Gamma)^*$  be a set of representatives of the  $\Gamma$ -inequivalent  $d$ -dimensional Voronoi cells that meet the interior  $X_n$ , and let  $\Sigma_d := \Sigma_d(\Gamma)$  be the subset of representatives of the  $\Gamma$ -inequivalent *orientable* cells in this dimension; here we call a cell *orientable* if all the elements in its stabilizer group preserve its orientation. Note that in our consideration the prime 2 will always be inverted. This entails that only orientable cells can contribute to the homology. One can form a chain complex  $\mathrm{Vor}_*$ , the *Voronoi complex*, and one can prove that modulo small primes the homology of this complex is the homology  $H_*(\Gamma, St_n)$ , where  $St_n$  is the rank  $n$  *Steinberg module* (cf. [4, p. 437]). To keep track of these small primes explicitly, we make the following definition.

**Definition 2.2** (Serre class of small prime power groups). *Given  $k \in \mathbb{N}$ , we let  $\mathcal{S}_{p \leq k}$  denote the **Serre class** of finite abelian groups  $G$  whose cardinality  $|G|$  has all of its prime divisors  $p$  satisfying  $p \leq k$ .*

*For any finitely generated abelian group  $G$ , there is a unique maximal subgroup  $G_{p \leq k}$  of  $G$  in the Serre class  $\mathcal{S}_{p \leq k}$ . We say that two finitely generated abelian groups  $G$  and  $G'$  are **equivalent modulo  $\mathcal{S}_{p \leq k}$**  and write  $G \simeq_{p \leq k} G'$  if the quotients  $G/G_{p \leq k} \cong G'/G'_{p \leq k}$  are isomorphic.*

We call the **torsion primes** of a group  $G$  those prime numbers  $p$  which divide the order of at least one of the finite subgroups of  $G$ .

**2.1. Voronoi data for  $R = \mathbb{Z}[i]$ .** We now give results for the Voronoi complexes and the equivariant homology of the pairs  $(X_n^*, \partial X_n^*)$  in the cases relevant to our paper ( $n = 2, 3, 4$ ). This subsection treats the Gaussian integers; in §2.2 we treat the Eisenstein integers.

**Proposition 2.3** ([20]).

1. *There is one  $d$ -dimensional Voronoi cell for  $\mathrm{GL}_2(\mathbb{Z}[i])$  for each  $1 \leq d \leq 3$ , and only the 3-dimensional cell is orientable.*
2. *The number of  $d$ -dimensional Voronoi cells for  $\mathrm{GL}_3(\mathbb{Z}[i])$  is given by:*

$d$	2	3	4	5	6	7	8
$ \Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i]))^* $	2	3	4	5	3	1	1
$ \Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i])) $	0	0	1	4	3	0	1

**Proposition 2.4** ([6, Table 12]). *The number of  $d$ -dimensional Voronoi cells for  $\mathrm{GL}_4(\mathbb{Z}[i])$  is given by:*

$d$	3	4	5	6	7	8	9	10	11	12	13	14	15
$ \Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i]))^* $	4	10	33	98	258	501	704	628	369	130	31	7	2
$ \Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i])) $	0	0	5	48	189	435	639	597	346	120	22	2	2

We remark that for  $\mathrm{GL}_3(\mathbb{Z}[i])$  the Voronoi complexes and their homology ranks were originally computed by Staffeldt [20], who even distilled the 3-part for each homology group. After calculating the differentials for this complex one obtains the following homology groups, in agreement with Staffeldt's results:

**Proposition 2.5** ([20, Theorems IV, 1.3 and 1.4, p.785]).

$$(1) \quad H_m(\mathrm{GL}_2(\mathbb{Z}[i]), St_2) \simeq_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(2) \quad H_m(\mathrm{GL}_3(\mathbb{Z}[i]), St_3) \simeq_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, from the above theorem we deduce that the only possible torsion primes for  $H_m(\mathrm{GL}_n(\mathbb{Z}[i]), St_n)$  for  $n = 2, 3$  are the primes 2 and 3.

While the Voronoi homology of  $\mathrm{GL}_4(\mathbb{Z}[i])$  has been determined in all degrees in [6, Theorem 7.2], we will only need the following two special cases.

**Proposition 2.6** ([6, Theorem 7.2]). *For  $m = 1, 2$  we have*

$$(3) \quad H_m(\mathrm{GL}_4(\mathbb{Z}[i]), St_4) \simeq_{/p \leq 5} \{0\}.$$

The last column of [6, Table 12] further shows that the elementary divisors of all the differentials in the Voronoi complex for  $\mathrm{GL}_4(\mathbb{Z}[i])$  in small degree (in fact for degree  $\leq 13$ ) are supported on primes  $\leq 3$ .

We want to show the stronger result that  $H_1(\mathrm{GL}_4(\mathbb{Z}[i]), St_4) \simeq_{/p \leq 3} \{0\}$ , i.e. we want to show that the prime 5 cannot occur. For this we will need to use spectral

sequences. According<sup>1</sup> to [5, VII.7], there is a spectral sequence  $E_{d,q}^r$  converging to the equivariant homology groups  $H_{d+q}^\Gamma(X_n^*, \partial X_n^*; \mathbb{Z})$  of the homology pair  $(X_n^*, \partial X_n^*)$ , and such that

$$(4) \quad E_{d,q}^1 = \bigoplus_{\sigma \in \Sigma_d^*} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma),$$

where  $\mathbb{Z}_\sigma$  is the orientation module of the cell  $\sigma$  and  $\Gamma_\sigma$  the stabilizer of the cell  $\sigma$ . In the remainder of this section we put  $n = 4$  and consider  $(X_4^*, \partial X_4^*)$ .

**Proposition 2.7.** *Let  $\Gamma = \mathrm{GL}_4(\mathbb{Z}[i])$  and  $E_{d,q}^1$  as above.*

(i) *For each  $d = 0, \dots, 4$  one has  $E_{d,4-d}^1 \simeq_{/p \leq 3} \{0\}$ .*

(ii) *Similarly, for each  $d = 0, \dots, 5$  one has  $E_{d,5-d}^1 \simeq_{/p \leq 3} \{0\}$ .*

*Proof.* We use the data obtained in [6, Table 12], available at [24].

(i) 1. As there are no cells in  $\Sigma_d^*$  for  $d \leq 2$ , we have  $E_{0,4}^1 = E_{1,3}^1 = E_{2,2}^1 = 0$ .

2. Consider now  $d = 3$ . The stabilizer of each of the four cells in  $\Sigma_3^*$  lies in  $\mathcal{S}_{p \leq 3}$ . Thus in particular we have

$$E_{3,1}^1 = \bigoplus_{\sigma \in \Sigma_3^*} H_1(\mathrm{Stab}_\sigma, \mathbb{Z}_\sigma) \in \mathcal{S}_{p \leq 3},$$

where  $\mathcal{S}_{p \leq 3}$  is as in Definition 2.2.

3. For  $d = 4$ , we note that none of the ten cells in  $\Sigma_4^*$  has its orientation preserved under the action of its stabilizer, so  $E_{4,0}^1 = 0 \bmod \mathcal{S}_2$ .

(ii) 1. As there are no cells in  $\Sigma_d^*$  for  $d \leq 2$ , we have  $E_{0,5}^1 = E_{1,4}^1 = E_{2,3}^1 = 0$ .

2. Consider now  $d = 3$  and  $d = 5$ . The stabilizer of each of the four cells in  $\Sigma_3^*$  and each of the 33 cells in  $\Sigma_5^*$  lies in  $\mathcal{S}_{p \leq 3}$ . Thus in particular we have

$$E_{3,2}^1 \in \mathcal{S}_{p \leq 3}, \quad E_{5,0}^1 \in \mathcal{S}_{p \leq 3}.$$

3. Finally, for  $d = 4$ , there is only one cell (out of ten) in  $\Sigma_4^*$ , denoted by  $\sigma_4^1$ , that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in  $H_1(\mathrm{Stab}(\sigma_4^1), \tilde{\mathbb{Z}})$  (where  $\tilde{\mathbb{Z}}$  is the orientation module  $\mathbb{Z}_{\sigma_4^1}$ ). Indeed, the subgroup  $K_1$  of  $\mathrm{Stab}(\sigma_4^1)$  preserving the orientation of  $\sigma_4^1$  is isomorphic to  $\mathbb{Z}/4\mathbb{Z} \times A_5$ , where  $A_5$  is the alternating group on five letters, with abelianization  $H_1(\mathrm{Stab}(\sigma_4^1), \tilde{\mathbb{Z}}) \simeq H_1(K_1, \mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$  (for the first equality, which holds mod  $\mathcal{S}_2$ , we make use of Lemmas 8.2 and 8.3 in [8]) lies in  $\mathcal{S}_{p \leq 3}$ . Thus there can be no 5-torsion from here, which completes the proof.  $\square$

**Corollary 2.8.** *For  $\Gamma = \mathrm{GL}_4(\mathbb{Z}[i])$  one has  $H_1(\Gamma, St_4) \simeq H_4^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p \leq 3} \{0\}$  and  $H_2(\Gamma, St_4) \simeq H_5^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p \leq 3} \{0\}$ .*

**2.2. Voronoi homology data for  $R = \mathbb{Z}[\rho]$ .** Now we turn to the Eisenstein case.

**Proposition 2.9** ([6, Tables 1 and 11]).

1. *There is one  $d$ -dimensional Voronoi cell for  $\mathrm{GL}_2(\mathbb{Z}[\rho])$  for each  $1 \leq d \leq 3$ , and only the 3-dimensional cell is orientable.*

<sup>1</sup>More precisely [5, VII.7] constructs a spectral sequence converging to the equivariant homology  $H_*^G(X, M)$  of a  $G$ -complex  $X$  with coefficients in a  $G$ -module  $M$ ; the  $E^1$  page has a form similar to (4). One can formulate an analogous spectral sequence for the equivariant homology of a pair  $(X, Y)$  of  $G$ -complexes with  $E^1$  page (4), cf. the remarks in [5, VII.7] in the paragraphs preceding equation (7.1).

2. The number of  $d$ -dimensional Voronoi cells for  $\mathrm{GL}_3(\mathbb{Z}[\rho])$  is given by:

$d$	2	3	4	5	6	7	8
$ \Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho]))^* $	1	2	3	4	3	2	2
$ \Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho])) $	0	0	1	2	1	1	2

3. The number of  $d$ -dimensional Voronoi cells for  $\mathrm{GL}_4(\mathbb{Z}[\rho])$  is given by:

$d$	3	4	5	6	7	8	9	10	11	12	13	14	15
$ \Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho]))^* $	2	5	12	34	82	166	277	324	259	142	48	15	5
$ \Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho])) $	0	0	0	8	50	129	228	286	237	122	36	10	5

After calculating the differentials we find the same results as for the homology of  $\mathbb{Z}[i]$  above:

**Proposition 2.10** ([6, Theorems 7.1 and 7.2 with Propositions 3.2 and 3.6]).

$$(5) \quad H_m(\mathrm{GL}_2(\mathbb{Z}[\rho]), St_2) \simeq_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(6) \quad H_m(\mathrm{GL}_3(\mathbb{Z}[\rho]), St_3) \simeq_{/p \leq 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise,} \end{cases}$$

For  $m = 1, 2$  we have

$$(7) \quad H_m(\mathrm{GL}_4(\mathbb{Z}[\rho]), St_4) \simeq_{/p \leq 5} \{0\}.$$

As with  $\mathbb{Z}[i]$ , a more refined analysis of the  $\Gamma = \mathrm{GL}_4(\mathbb{Z}[\rho])$  case shows that  $H_m^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z})$  contains no 5-torsion for  $m = 4, 5$ :

**Proposition 2.11.** Let  $\Gamma = \mathrm{GL}_4(\mathbb{Z}[\rho])$  and  $E_{d,q}^1$  as above.

(i) For each  $d = 0, \dots, 4$  one has  $E_{d,4-d}^1 \simeq_{/p \leq 3} \{0\}$ .

(ii) Similarly, for each  $d = 0, \dots, 5$  one has  $E_{d,5-d}^1 \simeq_{/p \leq 3} \{0\}$ .

*Proof.* The argument is very similar to that of the proof of Proposition 2.7. We use the data obtained in [6, Table 11], available at [24].

(i) 1. As there are no cells in  $\Sigma_d^*$  for  $d \leq 2$ , we have  $E_{0,4}^1 = E_{1,3}^1 = E_{2,2}^1 = 0$ .

2. For  $d = 3$ , there are two cells in  $\Sigma_3^*$ , with stabilizer in  $\mathcal{S}_{p \leq 3}$ , and hence

$$E_{3,1}^1 = \bigoplus_{\sigma \in \Sigma_3^*} H_1(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \in \mathcal{S}_{p \leq 3}.$$

3. For  $d = 4$ , we note that none of the five cells in  $\Sigma_4^*$  has its orientation preserved under the action of its stabilizer, so  $E_{4,0}^1 = 0 \bmod \mathcal{S}_2$ .

(ii) 1. As there are no cells in  $\Sigma_d^*$  for  $d \leq 2$ , we have  $E_{0,5}^1 = E_{1,4}^1 = E_{2,3}^1 = 0$ .

2. Consider now  $d = 3$  and  $d = 5$ . The stabilizer of each of the two cells in  $\Sigma_3^*$  and each of the 12 cells in  $\Sigma_5^*$  lies in  $\mathcal{S}_{p \leq 3}$ . Thus in particular we have

$$E_{3,2}^1 \in \mathcal{S}_{p \leq 3}, \quad E_{5,0}^1 \in \mathcal{S}_{p \leq 3}.$$

3. Finally, for  $d = 4$ , there is only one cell (out of five) in  $\Sigma_4^*$ , denoted by  $\sigma_4^1$ , that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in  $H_1(\mathrm{Stab}(\sigma_4^1), \tilde{\mathbb{Z}})$  (where  $\tilde{\mathbb{Z}}$  is the orientation module  $\mathbb{Z}_{\sigma_4^1}$ ). Indeed, the subgroup

$K_1$  of  $\text{Stab}(\sigma_4^1)$  preserving the orientation of  $\sigma_4^1$  is isomorphic to  $\mathbb{Z}/6\mathbb{Z} \times A_5$ , where  $A_5$  is the alternating group on five letters, with abelianization  $H_1(\text{Stab}(\sigma_4^1), \mathbb{Z}) = H_1(K_1, \mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z}$ , which lies in  $\mathcal{S}_{p \leq 3}$ . Thus there can be no 5-torsion from here, which completes the proof.  $\square$

**Corollary 2.12.** *For  $\Gamma = \text{GL}_4(\mathbb{Z}[\rho])$  one has  $H_1(\Gamma, St_4) \simeq H_4^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p \leq 3} \{0\}$  and  $H_2(\Gamma, St_4) \simeq H_5^\Gamma(X_4^*, \partial X_4^*, \mathbb{Z}) \simeq_{/p \leq 3} \{0\}$ .*

### 3. VANISHING AND SANDWICHING

In this section, we carry out the sandwiching argument (step (iv) of §1.2). As a first step we invoke a vanishing result for homology groups for  $BQ_1$  due to Quillen [15, p.212]. In our cases this result boils down to the following statement:

**Proposition 3.1.** *For the rings  $R = \mathbb{Z}[i]$  and  $\mathbb{Z}[\rho]$ , we have*

$$H_n(BQ_1(R)) = 0 \quad \text{whenever } n \geq 3.$$

For  $R = \mathbb{Z}[i]$  a slightly stronger result is proved in [20, Lemma I.1.2]. However, we will not need this stronger result for  $\mathbb{Z}[i]$ , or its analogue for  $\mathbb{Z}[\rho]$ .

Using our homology data from §2 and Proposition 3.1, we can get for both rings  $R = \mathbb{Z}[i]$  and  $R = \mathbb{Z}[\rho]$  the following result:

**Proposition 3.2.**  $H_5(BQ(R)) \simeq_{/p \leq 3} \mathbb{Z}$ .

*Proof.* For brevity we will drop  $R$  from the notation, as the argument is the same for both cases. We will successively determine  $H_5(BQ_j)$  for  $j = 1, \dots, 5$  and then identify the last group via stability with  $H_5(BQ)$ . For this, we will combine results from §2 with Quillen's long exact sequence for different  $j$ , given by

$$(8) \quad \cdots \longrightarrow H_n(BQ_{j-1}) \longrightarrow H_n(BQ_j) \longrightarrow H_{n-j}(\text{GL}_j, St_j) \longrightarrow H_{n-1}(BQ_{j-1}) \longrightarrow \cdots$$

*The case  $j = 1$ .* By Proposition 3.1 we have  $H_n(BQ_1) = 0$  for  $n \geq 3$ .

*The case  $j = 2$ .* From the above sequence (8) for  $j = 2$ , we get

$$\underbrace{H_5(BQ_1)}_{=0} \longrightarrow H_5(BQ_2) \longrightarrow H_3(\text{GL}_2, St_2) \longrightarrow \underbrace{H_4(BQ_1)}_{=0},$$

whence  $H_5(BQ_2) = 0 \pmod{\mathcal{S}_{p \leq 3}}$  by (1) and (5).

*The case  $j = 3$ .* Now we invoke another result of Staffeldt, who showed (see [20, proof of Theorem I.1.1]) that

$$(9) \quad H_4(BQ_2) = H_4(BQ_3) = \mathbb{Z} \pmod{\mathcal{S}_{p \leq 3}}.$$

From (8) for  $j = 3$  we get the exact sequence, working mod  $\mathcal{S}_{p \leq 3}$ ,

$$H_5(BQ_2) \longrightarrow H_5(BQ_3) \longrightarrow \underbrace{H_2(\text{GL}_3, St_3)}_{=\mathbb{Z} \text{ (by (2), (6))}} \longrightarrow \underbrace{H_4(BQ_2)}_{=\mathbb{Z} \text{ (by (9))}} \longrightarrow \underbrace{H_4(BQ_3)}_{=\mathbb{Z} \text{ (by (9))}} \longrightarrow \underbrace{H_1(\text{GL}_3, St_3)}_{=0 \text{ (by (2), (6))}}.$$

Since the leftmost group  $H_5(BQ_2)$  vanishes modulo  $\mathcal{S}_{p \leq 3}$  by the case  $j = 2$ , this sequence implies that  $H_5(BQ_3) = \mathbb{Z} \pmod{\mathcal{S}_{p \leq 3}}$ .

*The case  $j = 4$ .* Moreover, since  $H_2(\text{GL}_4, St_4) = H_1(\text{GL}_4, St_4) = 0 \pmod{\mathcal{S}_{p \leq 3}}$  by Proposition 2.6 and Propositions 2.7 and 2.11, the sequence (8) for  $j = 4$  gives in a similar way that

$$(10) \quad H_5(BQ_4) = H_5(BQ_3) = \mathbb{Z} \pmod{\mathcal{S}_{p \leq 3}}.$$

The case  $j = 5$ . This is the most complicated of all the cases to handle. Note that  $BQ$  is an  $H$ -space which implies that  $H_*(BQ) \otimes \mathbb{Q}$  is the enveloping algebra of  $\pi_*(BQ) \otimes \mathbb{Q}$ . It is well-known that  $K_0(\mathbb{Z}[i]) = \mathbb{Z}$ ,  $K_1(\mathbb{Z}[i]) = \mathbb{Z}/2$  and  $K_2(\mathbb{Z}[i]) = 0$  [3, Appendix] as well as  $K_3(\mathbb{Z}[i]) = \mathbb{Z} \oplus \mathbb{Z}/24$  (given by Merkurjev–Suslin, cf. e.g. Weibel [23], Theorem 73 in combination with Example 28), so modulo  $\mathcal{S}_{p \leq 3}$  we have

$$\pi_1(BQ) \otimes \mathbb{Q} = K_0(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q},$$

as well as  $\pi_2(BQ) \otimes \mathbb{Q} = \pi_3(BQ) \otimes \mathbb{Q} = 0$ , and

$$\pi_4(BQ) \otimes \mathbb{Q} = K_3(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q}.$$

A very similar argument works for  $\mathbb{Z}[\rho]$ .

Hence  $H_5(BQ) \otimes \mathbb{Q}$  contains the product of  $\pi_1(BQ) \otimes \mathbb{Q}$  by  $\pi_4(BQ) \otimes \mathbb{Q}$  and so its dimension is at least 1.

The stability result foreshadowed in step (iii) of §1.2 (resulting for a Euclidean domain  $\Lambda$  from  $H_0(\mathrm{GL}_n(\Lambda), St_n) = 0$  for  $n \geq 3$ , [13, Corollary to Theorem 4.1]), now implies that one has  $H_5(BQ) = H_5(BQ_5)$ . By the above we get that the rank of  $H_5(BQ_5) = H_5(BQ)$  is at least 1.

Therefore, invoking yet again Quillen’s exact sequence (8), this time for  $j = 5$ , and using the above result that  $H_5(BQ_4)$  is equal to  $\mathbb{Z}$  modulo  $\mathcal{S}_{p \leq 3}$ , we deduce from

$$\underbrace{H_5(BQ_4)}_{=\mathbb{Z} \text{ by (10)}} \longrightarrow H_5(BQ_5) \longrightarrow \underbrace{H_0(\mathrm{GL}_5, St_5)}_{=0}$$

that  $H_5(BQ) = H_5(BQ_5)$  must be equal to  $\mathbb{Z}$  modulo  $\mathcal{S}_{p \leq 3}$  as well. Thus  $H_5(BQ)$  cannot contain any  $p$ -torsion with  $p > 3$ .  $\square$

#### 4. RELATING $K_4(R)$ AND $H_5(BQ(R))$ VIA THE HUREWICZ HOMOMORPHISM

It is well known that for a number ring  $R$  the space  $BQ(R)$  is an infinite loop space. Hence a theorem due to Arlettaz [1, Theorem 1.5] shows that the kernel of the corresponding Hurewicz homomorphism  $K_4(R) = \pi_5(BQ) \rightarrow H_5(BQ)$  is certainly annihilated by 144 (cf. Definition 1.3 in loc.cit., where this number is denoted  $R_5$ ). Thus that kernel lies in  $\mathcal{S}_{p \leq 3}$  (Definition 2.2).

Therefore this Hurewicz homomorphism is injective modulo  $\mathcal{S}_{p \leq 3}$ . For  $R = \mathbb{Z}[i]$  or  $\mathbb{Z}[\rho]$ , Proposition 3.2 implies that  $H_5(BQ)$  contains no  $p$ -torsion for  $p > 3$ . After invoking Quillen’s result that  $K_{2n}(R)$  is finitely generated and Borel’s result that the rank of  $K_{2n}(R)$  is zero for any number ring  $R$  and  $n > 0$ , we obtain the following theorem:

**Theorem 4.1.** *The groups  $K_4(\mathbb{Z}[i])$  and  $K_4(\mathbb{Z}[\rho])$  lie in  $\mathcal{S}_{p \leq 3}$ .*

**Acknowledgments.** We thank Ph. Elbaz-Vincent for very helpful discussions. We also thank an anonymous referee for suggesting numerous improvements and corrections to our paper. This research was conducted as part of a “SQuARE” (Structured Quartet Research Ensemble) at the American Institute of Mathematics in Palo Alto, California in September 2013. It is a pleasure to thank AIM and its staff for their support, without which our collaboration would not have been possible.

## REFERENCES

- [1] D. Arlettaz, *The Hurewicz homomorphism in algebraic K-theory*, J. Pure Appl. Algebra **71** (1991), no. 1, 1–12.
- [2] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze.
- [3] H. Bass and J. Tate, *The Milnor ring of a global field*, Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972), Springer, Berlin, 1973, pp. 349–446. Lecture Notes in Math., Vol. 342.
- [4] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.
- [5] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.
- [6] M. Dutour Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, J. Pure Appl. Algebra **220** (2016), no. 7, 2564–2589.
- [7] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Quelques calculs de la cohomologie de  $GL_N(\mathbb{Z})$  et de la K-théorie de  $\mathbb{Z}$* , C. R. Math. Acad. Sci. Paris **335** (2002), no. 4, 321–324.
- [8] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math. **245** (2013), 587–624.
- [9] C. Haesemeyer and C. A. Weibel, *The Norm Residue Theorem in Motivic Cohomology*, in preparation.
- [10] M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen I*, Math. Ann. **141** (1960), 384–432.
- [11] M. Kolster, *Higher relative class number formulae*, Math. Ann. **323** (2002), no. 4, 667–692.
- [12] R. Lee and R. H. Szczarba, *The group  $K_3(\mathbb{Z})$  is cyclic of order forty-eight*, Ann. of Math. (2) **104** (1976), no. 1, 31–60.
- [13] R. Lee and R. H. Szczarba, *On the homology and cohomology of congruence subgroups*, Invent. Math. **33** (1976), no. 1, 15–53.
- [14] R. Lee and R. H. Szczarba, *On the torsion in  $K_4(\mathbb{Z})$  and  $K_5(\mathbb{Z})$* , Duke Math. J. **45** (1978), no. 1, 101–129.
- [15] D. Quillen, *Finite generation of the groups  $K_i$  of rings of algebraic integers*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 179–198. Lecture Notes in Math., Vol. 341.
- [16] D. Quillen, *Higher algebraic K-theory I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.
- [17] J. Rognes,  *$K_4(\mathbb{Z})$  is the trivial group*, Topology **39** (2000), no. 2, 267–281.
- [18] C. Soulé, *On the 3-torsion in  $K_4(\mathbb{Z})$* , Topology **39** (2000), no. 2, 259–265.
- [19] C. Soulé, *The cohomology of  $SL_3(\mathbb{Z})$* , Topology **17** (1978), no. 1, 1–22.
- [20] R. E. Staffeldt, *Reduction theory and  $K_3$  of the Gaussian integers*, Duke Math. J. **46** (1979), no. 4, 773–798.
- [21] V. Voevodsky, *On motivic cohomology with  $\mathbb{Z}/l$ -coefficients*, Ann. of Math. (2) **174** (2011), no. 1, 401–438.
- [22] G. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques I: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math **133** (1908), no. 1, 97–178.
- [23] C. Weibel, *Algebraic K-theory of rings of integers in local and global fields*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 139–190.
- [24] D. Yasaki, *Voronoi tessellation data*, available at [www.uncg.edu/mat/faculty/d\\_yasaki/data/k4imquad/](http://www.uncg.edu/mat/faculty/d_yasaki/data/k4imquad/).

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