Open Problems collected from the SPRING TOPOLOGY AND DYNAMICAL SYSTEMS CONFERENCE 2006

Edited by Jerry E, Vaughan

The fortieth annual Spring Topology and Dynamical Systems Conference was held March 23–25, 2006 at the University North Carolina at Greensboro. The conference[1] featured six plenary talks, twelve semi-plenary talks, and four parallel special in the areas of Continuum Theory, Dynamical Systems, General/Set-Theoretic Topology, and Geometric Topology and Geometric Group Theory.

We have collected the following remarks and open problems contributed by speakers at the conference, and organized them by the titles of the four special sessions. In some cases our organization of these contributions is somewhat arbitrary because of the interaction between the various areas at the conference as was evident by the large attendance at the plenary and semi-plenary talks, and from the makeup of the audiences at many of the talks in the special sessions.

Information about the conference, including abstracts of the talks, is available at Topology Atlas:

http://at.yorku.ca/cgi-bin/amca-calendar/d/facv72

1. Continuum Theory

1.1. **Charles L Hagopian.** My lecture focused on fixed-point results that followed and in many cases were motivated by Bing's 1969 expository article, "The elusive fixed-point property." At the center of this area is the problem of determining whether every plane continuum that does not separate the plane has the fixed-point property. Bellamy's 1979 example of a tree-like continuum without the fixed-point property has given us insight into the nature of this classical problem. As I stated in my lecture, the problem would be solved if one could embed Bellamy's second example (defined by applying the Fugate-Mohler technique to Bellamy's first example) in the plane. It would also be a major breakthrough to prove every triod-like continuum has the fixed-point property. Recent examples of Sobolewski, Prajs, and myself, which answer questions of Bing, cause us to believe there exists a plane continuum with the fixed-point property whose product with an interval does not have the fixed-point property. This is another unsolved problem of Bing. The most recent result that I stated in my lecture

[1]The conference organizers gratefully acknowledge support for the conference from NSF Grant DMS 0539088, and from both the College of Arts and Sciences and the Department of Mathematical Sciences, UNCG.

is Illanes's beautiful example of a tree-like continuum (a spiral to a triod) whose cone admits a fixed-point-free map. It is not known if the cone over a uniquely arcwise connected plane continuum must have the fixed-point property. Illanes's example can be modified to show the answer to this question is no for uniquely arcwise connected continuum in Euclidean 3-space.

1.2. **Sergio Macias.** Questions and Problems by the late Professor Janusz J. Charatoinik:

A *continuum* is a nonempty compact, connected, metric space. A continuum X is said to be *decomposable* if it is the union of two of its proper subcontinua. The continuum X is *indecomposable* if it is not decomposable. The continuum X is *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its nondegenerate subcontinua is decomposable (indecomposable).

A *dendroid* is an arcwise connected continuum such that the intersection of any two subcontinua is connected. A *dendrite* is a locally connected dendroid.

By a *map* we mean a continuous function. A surjective map $f\colon X \to Y$ between continua is said to be:

(i) *open* provided for each open subset U of X, $f(U)$ is open in Y.

(ii) *monotone* if $f^{-1}(y)$ is connected for every $y \in Y$.

(iii) *light* provided that $f^{-1}(y)$ is totally disconnected for each $y \in Y$.

Given a continuum X, the *hyperspaces* of X are:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\};$$

$$C_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ compontents}\};$$

$$F_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}.$$

We topologize these sets with the *Hausdorff metric* defined by:

$$H(A, B) = \inf\{\varepsilon > 0 \mid A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\}$$

where $N(\varepsilon, A)$ is the ε-open ball about A.

Remark. In the literature $C_1(X)$ is denoted by $C(X)$. The interest on $C_n(X)$ is recent.

Characterization of dendrites is one of the oldest problems in the study of dendroids. In [4] there are over 60 equivalent definitions of dendrites. Some of these definitions are in terms of maps. A problem from [4] is:

(1) Characterize all dendrites X having the property that each open image of X is homeomorphic to X. [4, Problem 2.14]

Professor Charatonik proved the following result:

Theorem 1. Let D be a dendrite. For any compact space X and for any light open map $f\colon X \to Y$, where $D \subset Y$, there exists a homeomorphic copy $D' \subset X$ of D such that $f|_{D'}\colon D' \to D$ is a homeomorphism. [1]

Motivated by Theorem 1, we have the following problem:

(2) Characterize all dendrites X having the property if a dendrite Z can be mapped onto X by a monotone map, then Z contains a homeomorphic copy of X. [1, Problem 1.3]

Professor Charatonik was the first person interesting in *generalized homogeneity*. Let $\mathcal{M}$ be a class of surjective maps of continua. We say that a continuum X is *homogeneous with respect to* $\mathcal{M}$ provided that for any two points x_1 and x_2 of X there exists $f \in \mathcal{M}$ such that $f(x_1) = x_2$.

(3) What dendrites are homogeneous with respect to monotone maps? [2, Question 7.2]

A continuum X has the *property of Kelley* provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two points $a, b \in X$ with $d(a, b) < \delta$ and any $A \in C(X)$, there exists $B \in C(X)$ such that $b \in B$ and $H(A, B) < \varepsilon$.

The property of Kelley has been proved to be an important one. For example, if a continuum X has the property of Kelley, then 2^X and $C_n(X)$ are contractible. Professor Charatonik asked:

(4) For what continua X does the property of Kelley imply local connectedness of X at some point? [3, Question 5.20]

Regarding hyperspaces, a geometric way to see hyperspaces is as a "cone", even though it is not true that all hyperspaces are homeomorphic to cones, they have a lot of similarities and there are some cases in which a hyperspace is homeomorphic to a cone. With this in mind, Professors Nadler and Macías have the following questions:

(5) Does there exist a continuum X, that is not an arc, for which there is an integer $n \geq 2$ such that $C_n(X)$ is homeomorphic to the product of two finite-dimensional continua? [6, Question 4.12]

(6) Does there exist an indecomposable continuum X such that $C_n(X)$ is homeomorphic to the cone over a finite-dimensional continuum for some $n \geq 2$? [6, Question 3.7]

(7) Does there exist a hereditarily decomposable continuum X such that is neither an arc nor a simple m-od such that $C_n(X)$ is homeomorphic to the cone over a finite-dimensional continuum for some $n \geq 2$? [5, Question 3.3]

References

[1] J. J. Charatonik, Dendrites and monotone mappings, Math. Pannon. 15 (2004), 115-125.

[2] J. J. Charatonik, Monotone mappings of universal dendrites, Topology Appl. 38 (1991) 163-187.

[3] J. J. Charatonik, On acyclic curves. A survey of results and problems, Bol. Soc. Mat. Mexicana, (3) 1 (1995), 1-39.

[4] J. J. Charatonik and W. J. Charatonik, Dendrites, Aportaciones Matemáticas, Comunicaciones # 22, Sociedad Matemática Mexicana, (1998), 227-253.

[5] S. Macías, Fans whose hyperspaces are cones, Topology Proc., 27 (2003), 217-222.

[6] S. Macías and S. B. Nadler, Jr., n-fold hyperspaces, cones and products, Topology Proc., 26 (2001-2002), 255-270.

1.3. **Patricia Pellicer-Covarrubias.** Let $m \in \mathbb{N}$. We say that a continuum X is $\frac{1}{m}$-*homogeneous* provided that the action of the group of homeomorphisms of X onto itself has exactly m orbits. A continuum is *indecomposable* provided it cannot be expressed as the union of two of its proper subcontinua.

Problem 1.1. Is there a $\frac{1}{2}$-homogeneous indecomposable arc-like (or circle-like) continuum?

It is known that if X is a 1-dimensional continuum, then $Cone(X)$ is $\frac{1}{2}$-homogeneous if and only if X is an arc or a simple closed curve. It is also known that this cannot be generalized to dimension $n \geq 4$ ([3]). Thus, we have the following question:

Problem 1.2. a) If X is a continuum of dimension $n = 2$ or 3 such that $Cone(X)$ is $\frac{1}{2}$-homogeneous, must X be an n-cell or an n-sphere? What about when X is locally connected?

b) If the cone over a finite-dimensional continuum is $\frac{1}{2}$-homogeneous, must the cone be an n-cell? What about when X is locally connected?

Other results and problems related to $\frac{1}{2}$-homogeneity on cones of continua can be found in [3].

For a continuum X, the *hyperspace* 2^X is the space of all closed, nonempty subsets of X with the Hausdorff metric. We also define: $C(X) = \{A \in 2^X : A \text{ is connected}\}$ and $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$.

Recent research ([1]) has shown that if there exists a positive integer k such that X does not contain k-ods, then $C_n(X)$ is $\frac{1}{2}$-homogeneous if and only if: i) $n = 1$ and X is an arc or a simple closed curve, or ii) $n = 2$ and X is an arc. Moreover, if X is locally connected, then $C_n(X)$ is $\frac{1}{2}$-homogeneous if and only if: i) $n = 1$ and X is an arc or a simple closed curve, or ii) $n = 2$ and X is an arc. The following question remains unanswered:

Problem 1.3. If X is a continuum such that $C_n(X)$ is $\frac{1}{2}$-homogeneous, then is X an arc or a simple closed curve?

We conclude noting that $\frac{1}{2}$-homogeneity has been of recent interest and we invite the reader to look for results and problems on the topic in [1], [2], [3], [4] and [5].

References

[1] A. Illanes, *Hyperspaces with Exactly Two or Three Orbits*, to appear in Aportaciones Matemáticas de la Sociedad Matemática Mexicana.

[2] S. B. Nadler, Jr., and P. Pellicer-Covarrubias, *Hyperspaces with exactly two orbits*, Glasnik Mat. **41**, no.1 (2006), 141-157.

[3] S. B. Nadler Jr., and P. Pellicer-Covarrubias, *Cones that are $\frac{1}{2}$-homogeneous*, Houston J. of Math. **33**, no.1 (2007), 229–247.

[4] S. B. Nadler, Jr., P. Pellicer-Covarrubias, I. Puga. *$\frac{1}{2}$-homogeneous continua with cut points,* to appear in Topology Appl.

[5] V. Neumann-Lara, P. Pellicer-Covarrubias and I. Puga, *On $\frac{1}{2}$-homogeneous continua*, Topology Appl. **153** (2006), 2518–2527.

2. Dynamical Systems

2.1. **Louis Block.** Let f be a continuous map of the interval I to itself. Let (I, f) denote the inverse limit space obtained from the inverse sequence all of whose maps are f and all of whose spaces are I. Suppose that f has a periodic point of period larger than one, and (I, f) is homeomorphic to the pseudoarc. Does it follow that f has periodic points of all periods?

Partial results appear in the paper of Block, Keesling, and Uspenskij, "Inverse limits which are the pseudoarc," Houston Journal of Mathematics, volume 26, #4, 2000, Pages 629 - 638.

2.2. **Henk Bruin.** Let me add some questions that are related to my own talk. They involve inverse limit spaces of unimodal maps, and not chaotic attractors as such. I see these inverse limit spaces as a step towards understanding the structure of, for example, strange Henon attractors. The questions below can, with minor changes in wording, be asked just as well for Henon attractors.

An inverse limit space of bonding map $f : X \to X$ on metric space X is the set of backward orbits $\{x = (x_0, x_1, x_2, \dots) : x_i = f(x_{i+1} \in X\}$ equipped with product topology. In general, they are continua (compact, connected, metric spaces) of a very intricate structure. Within dynamics, they play a role in describing chaotic attractors.

If $f : I \to I$ is an endomorphism of the interval (such as the logistic map $f(x) = ax(1 - x)$), a major question, attributed to Ingram, is whether two non-conjugate maps can have homomorphic inverse limit spaces. This question has been settled (Kailhofer, Stimac, Block et al.) only for maps with a finite critical omega-limit set, i.e. the set $\omega(c)$ of limit points of the orbit of the critical point c is finite.

More detailed questions are: 1) Is it true that every self-homeomorphism on the inverse limit space is homotopic to an iterate of the shift-transformation $\sigma(x_0, x_1, x_2, \dots) = (f(x_0), x_0, x_1, x_2, \dots)$? 2) Can one recapture dynamical features, such as the entropy of the bonding map, from the topological structure of the inverse limit space? 3) Are any pair of arc-composants (i.e. continuous bijective images of the real line within the inverse limit space homeomorphic to each other. If not, classify them. 4) It is shown (Barge and Diamond) that if c is periodic, then some collection of arc-composants A_k are asymptotic to each other, i.e. each A_k allows a parametrization $g_k : R \to A_k$ such that the distance $d(g_k(t), g_l(t)) \to 0$ as $t \to \infty$, and $k \neq l$. Are there any asymptotic arc-composants when c is not periodic, and if so, classify them.

One specific example, stemming from my paper (H Bruin, Asymptotic arc-components of unimodal inverse limit spaces, Top. Appl. 152 (2005) 182 - 200.) is the logistic map where c has period 3. In this case, there is a self-asymptotic arc-composant A, i.e. there is a parametrization $g : R \to A$ such that $d(g(t), g(-t)) \to 0$ as $t \to \infty$. This arc-composant would be a strong candidate for not being homeomorphic to another arc-composant in the space, but so far I haven't been able to prove it.

2.3. **Grzegorz Graff. Periodicity of indices of iterations for homeomorphisms**

Let $ind(f, x_0)$ be a local fixed point index at x_0 , where f is a selfmap of $\mathbb{R}^m$. Under the assumption that x_0 is an isolated fixed point for each f^n (i.e. for each n there is an isolation neighborhood U_n) $\{ind(f^n, 0)\}_{n=1}^{\infty}$ is well-defined.

The sequence of indices of iterations is a powerful device in periodic point theory. Its applications are specially fruitful if it is known that $\{ind(f^n, 0)\}_{n=1}^{\infty}$ is a periodic sequence.

Problem 1 Let f be a homeomorphism of $\mathbb{R}^m$. Assume that: $(*)$ there is a neighborhood U of x_0 such that there are no periodic orbit, except for x_0 , in U. Is that true that $\{ind(f^n, 0)\}_{n=1}^{\infty}$ is a periodic sequence?

Comment 1 Without the assumption $(*)$ this statement is true for $m = 1$ (in an obvious way) and $m = 2$ (cf. [2], [5]). It is false for$m \geq 3$ (cf. [1]).

Problem 2 What if we change the assumption $(*)$ in Problem 1 by the stronger condition: $(**)$ x_0 is not a repelling fixed point and there is a neighborhood U of x_0 such that $\bigcap_{k\in\mathbb{Z}} f^k(U) = \{x_0\}$ (i.e., $\{x_0\}$ is an isolated invariant set).

Comment 2 Except for periodicity, strong restrictions on the form of indices of iterations were found in this case for m = 2 (cf. [3], [4], [6]).

References

[1] I. K. BABENKO and S. A. BOGATYI, The behavior of the index of pe riodic points under iterations of a mapping, Math. USSR Izv. 38 (1992), 1-26.

[2] M. BROWN, On the fixed point index of iterates of planar homeomorphisms, Proc. Amer. Math. Soc. 108 (1990), no. 4, 1109-1114.

[3] P. CALVEZ and J.-C. YOCCOZ, Un th?eoreme dindice pour les hom?eomorphismes du plan au voisinage dun point fixe, Annals of Math. 146 (1997), 241-293.

[4] J. FRANKS, The Conley index and non-existence of minimal homeomorphisms, Proceedings of the Conference on Probability, Ergodic Theory, and Analysis (Evanston, IL, 1997), Illinois J. Math. 43 (1999), no. 3, 457-464.

[5] G. GRAFF and P. NOWAK-PRZYGODZKI, Fixed point indices of iterations of planar homeomorphisms, Topol. Methods Nonlinear Anal. 22 (2003), no. 1, 159-166.

[6] F. R. RUIZ DEL PORTAL and J. M. SALAZAR, Fixed point index of iterations of local homeomorphisms of the plane: a Conley index approach, Topology 41 (2002), no. 6, 1199-1212.

2.4. **Krystyna Kuperberg. Definitions.** A *dynamical system*, or an $\mathbb{R}$-*action*, on a metric space X is a continuous map $\Phi : \mathbb{R} \times X \to X$ such that $\Phi(0, p) = p$ and $\Phi(t + s, p) = \Phi(s, \Phi(t, p))$ for $p \in X$ and t and s in $\mathbb{R}$. A *trajectory* of a point p is the set $\Phi(\mathbb{R} \times \{p\})$. A point p whose trajectory consists of p is a *fixed point*. A *periodic trajectory* is a trajectory homeomorphic to S^1. The trajectories are *uniformly bounded* if the set of diameters is bounded. A set A is *invariant* if $p \in A$ implies $\Phi(\mathbb{R} \times \{p\}) \subset A$. An invariant set is *isolated* if there is neighborhood of A in which A is the largest invariant set.

If X is furnished with a measure, then Φ is *measure preserving* if for each $t \in \mathbb{R}$, the map $\Phi(t, p) : X \to X$ is measure preserving.

If X is a 3-manifold, then a trajectory of p is *wild* if the closure of $\Phi(\mathbb{R}^- \times \{p\})$ or the closure of $\Phi(\mathbb{R}^+ \times \{p\})$ is a wild arc.

A compact invariant set $A \subset X$ is *stable* if for every neighborhood U of A, there exists a neighborhood V of A such that $\{\Phi(t, p) \,|\, t \geq 0,\ p \in V\} \subset U$. A compact set A is *movable* in X if for every neighborhood U of A there exists a neighborhood V of A such that for every neighborhood W of A there is a homotopy $H : V \times [0, 1] \to U$ such that $H(p, 0) = p$ and $H(p, 1) \in W$ for $p \in V$. (Note that the definition of movability is unrelated to the dynamical system Φ.)

Questions.

(1) (Greg Kuperberg) Does there exist a fixed point free, measure preserving dynamical system on $\mathbb{R}^3$ with uniformly bounded trajectories? The question may be modified by requiring additional conditions such as: Φ has no periodic trajectories, or Φ is C^r $(r \geq 1)$ $[C^\infty, C^\omega]$.

Comment: A fixed point free, measure preserving C^0 $[C^1]$ dynamical system on $\mathbb{R}^3$ with uniformly bounded trajectories and with a discrete set of periodic trajectories can be modified to a fixed point free, measure preserving C^0 $[C^1]$ dynamical system with uniformly bounded trajectories and no periodic trajectories.

(2) Let M be a boundaryless 3-manifold.
   (a) Does there exist a C^∞ $[C^r$ $(r \geq 1)]$ dynamical system on M with a discrete set of fixed points and with every non-trivial trajectory wild?
   (b) If M is closed, does there exist a C^r $(r \geq 1)$ $[C^\infty]$ dynamical system on M with exactly one fixed point and with every non-trivial trajectory wild? In particular, does such a dynamical system exist on S^3?

   Comment: It is known that there exist dynamical systems as above such that the map Φ restricted to any of the sets $\{t\} \times M$ is C^∞, but Φ is only C^0.

(3) Does there exist a measure preserving dynamical system on $\mathbb{R}^3$ with a discrete set of fixed points and with every non-trivial trajectory wild?

(4) Let A be a compact set invariant under a dynamical system Φ on $\mathbb{R}^3$.
   (a) Does every neighborhood U of A contain a compact invariant movable set containing A?
   (b) Is A contained in a compact invariant movable set?
   (c) If A is 1-dimensional, does every neighborhood U of A contain a compact invariant movable 1-dimensional set containing A?
   (d) If A is 1-dimensional, then is A contained in a compact invariant movable 1-dimensional set?
   (e) If A is a solenoid, does every neighborhood of A intersect a periodic trajectory?
   (f) If A is a solenoid, does every neighborhood of A contain a periodic trajectory?

   Comments: If A is a stable solenoid, then every neighborhood of A contains a periodic trajectory. Otherwise not much is known about the topic. The questions can be modified by adding the assumption that Φ is fixed point free. Other modification is to replace $\mathbb{R}^3$ with the product $\mathbb{R}^2 \times S^1$ and restrict the class of dynamical systems to suspensions of planar orientation preserving homeomorphisms.

(5) Does there exist a C^3 $[C^\infty]$ dynamical system on S^3 with no periodic trajectories and every compact invariant set isolated?

2.5. **Michal Misiurewicz.** For a continuous semiflow Φ on a compact space X with a continuous observable cocycle ξ (that is, $\xi : [0,\infty) \times X \to \mathbb{R}^m$ and $\xi(t+s,x) = \xi(t,x) + \xi(s,\Phi^t(x))$), the *rotation set* R of (X,ϕ,ξ) consists of limits of the sequences $(\xi(t_n,x_n)/t_n)_{n=1}^{\infty}$, where t_n goes to infinity. By the definition, it is closed, andf if ξ is time-Lipschitz continuous then it is easy to prove that it is connected.

An *observable function* is a function $\zeta : X \to \mathbb{R}^m$ such that $\xi(t,x) = \int_0^t \zeta(\Phi^s(x))\, ds$. If the limit $\lim_{t\to\infty} \xi(t,x)/t$ exists, it is called the *rotation vector* of x (or of its orbit). It exists for all periodic orbits and for all generic points of ergodic measures (then it is the integral of the observable function).

Desirable properties of R are:

(a) Rotation set is convex.
(b) Rotation vectors of periodic orbits are dense in the rotation set.
(c) If $\vec{u}$ is a vector from the interior of R, then there exists a nonempty compact invariant subset Y of the phase space, such that every point from Y has rotation vector $\vec{u}$. Therefore, there exists an ergodic invariant probability measure on the phase space, for which the integral of the velocity is equal to $\vec{u}$.

An interesting example example where the rotation set can be considered is a billiard on the m-dimensional torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ ($m \geq 2$) with one or more obstacles with smooth boundaries. Then the phase space X is

$$(\overline{\mathbb{T}^m \setminus O}) \times S^{m-1}$$

with incoming and outgoing vectors on the boundary of the obstacle identified. The observable cocycle is the displacement in the lifting, and the observable function is the velocity.

If there is only one obstacle with strictly convex boundary and the diameter less than $\sqrt{2}/4$, we show in [Alexander Blokh, Michał Misiurewicz and Nándor Simányi, *Rotation sets of billiards with one obstacle*, Commun. Math. Phys., published online April 14, 2006] that there is a large subset $AR \subset R$ for which the properties (a)-(c) hold.

Question 2.1. *Under the above assumptions, do the properties (a)-(c) hold for the whole rotation set R?*

Question 2.2. *What other assumptions on the obstacles would yield the properties (a)-(c) for the whole rotation set R or for its substantial subset?*

3. General/Set-Theoretic Topology

3.1. **Raushan Buzyakova.** We consider only Tychonoff spaces. We say that a space X has a zero-set diagonal if the diagonal $\Delta_X = \{(x,x) : x \in X\}$ is a zero-set in X^2. A space X has a regular G_δ-diagonal if there exists a countable family $\{U_n\}_n$ of open neighborhoods of Δ_X in X^2 such that $\Delta_X = \cap_n \overline{U}_n$.

It is proved in [B] that if X has a zero-set diagonal and X^2 has countable extent and then X is submetrizable. This motivates the following questions.

Question. *Let X have a zero-set diagonal and countable extent. Is X submetrizable?*

Question. *Is there a non-submetrizable space X with a regular G_δ-diagonal such that X^2 has countable extent?*

Recall that a space X is ω_1-Lindelöf if every ω_1-sized open cover of X contains a countable subcover. It is known that the square of a Čech-complete ω_1-Lindelöf space is ω_1-Lindelöf [K]. This fact and the mentioned result imply that if a Čech-complete ω_1-Lindelöf space has a zero-set diagonal then it is submetrizable. It is known [N] that a paracompact Čech-complete space with a G_δ-diagonal is metrizable. This prompts the following question.

Question. *Let X be a Čech-complete ω_1-Lindelöf space with a zero-set diagonal. Is X metrizable? What if X is a p-space?*

Recall that a space X is linearly Lindelöf if every open cover of X that forms a chain contains a countable subcover. A slight modification of Sneider's theorem [S] states that a Lindelöf space with a G_δ-diagonal is submetrizable. So far it has been rather hard to distinguish Lindelöfness from linearly Lindelöfess. This motivates the following question.

Question. (A. V. Arhangelskii) *Let X be a linearly Lindelöf space with a G_δ-diagonal. Is X submetrizable?*

In Arhangelskii's question we do not know answers even if linear Lindelöfness is replaced by ω_1-Lindelöfness, or/and G_δ-diagonal is replaced by regular G_δ-diagonal/zero-set diagonal.

References

[B] Buzyakova R. Z., *Observations on spaces with zero-set or regular G_δ-diagonals*, Comment. Math. Univ. Carolinae, 46 3 (2005) 469-473.

[K] A. N. Karpov, *Countable products of Chech-complete linearly Lindelöf and initially-compact spaces*, Vestnik Mosk. Univ. Ser. I Mat-Mekh., 2000, no 5, 7–9, 67.

[S] V. Sneider, *Continuous images of Souslin and Borel sets; Metrization theorems*, Dokl. Acad. Nauk SSSR, 50, 1959, 77–79.

[N] J. Nagata, *On necessary and sufficient condition of metrizability*, J. Inst. Polyt. Osaka City Univ. 1(1950), 93-100.

3.2. **Gary Gruenhage.** The following are some questions that were raised (by the indicated speaker) at the conference.

(1) Is every paracompact (or Lindelöf, or submetacompact) space a D-space? (D. Burke)
(2) If X is compact Hausdorff and has a small diagonal, must X be metrizable? (A. Dow)
(3) Does every Tychonoff connected space have a strictly stronger connected topology? (W. Hu)

3.3. **Jan van Mill.** All spaces under discussion are separable and metrizable. A space is absolutely Borel if it is a Borel set of every space it is embedded in. A space is analytic if it is a continuous image of the space of irrational numbers. A space is Polish if it is topologically complete. A space is coanalytic if it can be embedded in a Polish space in such a way that its "remainder" is analytic.

Problem 1: Let X be a Polish space on which some (separable metrizable) group acts transitively. Is there a Polish group that acts transitively on X?

Problem 2: Let X be a Polish space on which some absolutely Borel group acts transitively. Is there a Polish group that acts transitively on X?

Problem 3: Let X be absolute Borel and assume that some (separable metrizable) group acts transitively on it. Is there an absolutely Borel group that acts transitively on X?

Problem 4: Let X he a coanalytic, homogeneous and strongly locally homogeneous space. Is there a coanalytic topological group that acts transitively on X?

3.4. **F. Javier Trigos-Arrieta.** All topological groups are Abelian and Tychonoff.

If H is a dense subgroup of the topological group G, then we say that H *determines* G if $\widehat{G}$ is topologically isomorphic to $\widehat{H}$, when both groups are equipped with the compact open topology. Then G is said to be *determined* if every dense of its subgroups determines G.

Question 1. *Assume that G_1 and G_2 are determined groups. Is $G_1 \times G_2$ determined?*

Yes when

(1) Both groups are metrizable (Aussenhofer [1], [2] and Chasco [4].)
(2) When one group is discrete (T-A, unpublished).

Unknown even when

(1) One group is metrizable.
(2) One group is compact.
(3) One group is compact and metrizable.

Reference: [5] and [12].

Question 2. *Assume that G is a compact group of weight w with $\aleph_1 \leq w < \mathfrak{c}$. Is G determined?*

Unknown even when

(1) $G = \mathbb{T}^{\aleph_1}$.
(2) $G = F^{\aleph_1}$, F finite.

Note: G is determined if its weight is $\aleph_0$ (equivalently, when G is metrizable, Aussenhofer [1], [2] and Chasco [4]). G is not determined if its weight is $\mathfrak{c}$ or bigger [5].

Reference: [5] and [12].

Question 3. *Is there a (measurable) subgroup A of $\mathbb{T}$ of cardinality $|A|$ with $\aleph_1 \leq |A| < \mathfrak{c}$ such that the only compact sets of $(\mathbb{Z}, \tau_A)$ are the finite ones?*

Never if $|A| = \aleph_0$. Yes if $|A| = \mathfrak{c}$. For example, Leptin [9] and Glicksberg [7] were the first ones to prove it when $A = \mathbb{T}$. Comfort, Trigos and Wu [6] proved it true whenever A is a non-measurable subgroup of $\mathbb{T}$. Barbieri, Dikranjan, Milan and Weber [3] (under MA) and Hart and Kunen [8] proved there exists A of measure zero such that the only compact sets of $(\mathbb{Z}, \tau_A)$ are the finite ones; their subgroups A have cardinality $\mathfrak{c}$. On the other hand, Raczkowski [10] and [11] proved the existence of families $\mathcal{A}$ of groups A_k, and $\mathcal{B}$ of groups A_B, each family of size $2^{\mathfrak{c}}$ and each of A_k and B_k, of cardinality $\mathfrak{c}$ such that

(1) the only compact sets of $(\mathbb{Z}, \tau_{A_k})$ are the finite ones,
(2) $(\mathbb{Z}, \tau_{B_k})$ has non-trivial convergent sequences.

Reference: [12].

References

[1] L. Außenhofer, *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups,* Ph.D. thesis. Universität Tübingen 1998.

[2] L. Außenhofer, *Contributions to the Duality Theory of Abelian Topological Groups and to the Theory of Nuclear Groups,* Dissertationes Math., **CCCLXXXIV** (1998), Warszawa.

[3] Barbieri, Dikranjan, Milan and Weber, *Answer to Raczkowski's questions on convergent sequences of integers,* Top. Appl., 132, 2003, 89-101.

[4] M. J. Chasco, *Pontryagin duality for metrizable groups*, Archiv der Math. **70** (1998), 22–28.

[5] W. W. Comfort, S. U. Raczkowski and F. Javier Trigos-Arrieta, *The dual group of a dense subgroup ,* Czechoslovak Mathematical Journal, 54 (129) (2004), 509-533. MR 2005h:22001. Zbl 1080.22500.)

[6] W. W. Comfort, F. Javier Trigos-Arrieta, and Ta-Sun Wu, *The Bohr compactification, modulo a metrizable subgroup*, Fund. Math. **143** (1993), 119–136, Correction: **152** (1997), 97–98.

[7] Irving Glicksberg, *Uniform boundedness for groups*, Canadian J. Math. **14** (1962), 269–276.

[8] Hart and Kunen. *Limits in function spaces and compact groups,* Top. Appl., 151, 2005, 157-168.

[9] Horst Leptin, *Abelsche Gruppen mit kompakten Charaktergruppen und Dualitätstheorie gewisser linear topologischer abelscher Gruppen*, Abhandlungen Mathem. Seminar Univ. Hamburg **19** (1955), 244–263.

[10] Sophia U. Raczkowski-Trigos, *Totally Bounded Groups*, Ph.D. thesis, Wesleyan University, Middletown, Connecticut, USA, 1998.

[11] S. Raczkowski, *Totally bounded topological group topologies on the integers*, Topology Appl., **121**, 2002.

[12] F. Javier Trigos-Arrieta, *Two problems in topological groups.* Talk given at the 40th Spring Topology and Dynamical Systems Conference, Greensboro, NC, March 23-25, 2006. At http://www.csub.edu/ jtrigos/research/sample-1.pdf .)

3.5. **Judith Roitman and Scott Williams.** We recall some long unsolved problems in the theory of box products.

Suppose X is a compact space and $\Pi = \Pi^\omega X$ is given the box topology.

(1) If X has weight at most ω_1, is Π normal?
(2) If X is first countable, is Π normal?
(3) If X is compact metric, is Π normal?
(4) If $X = [0, 1]$, is Π normal?
(5) If $X =$ the Cantor set, is Π normal?
(6) If $X = \omega + 1$, is Π normal?

Note: yes to (1) or (2) yields (3) and the rest. $\mathfrak{d} = \omega_1$ implies (1). $\mathfrak{d} = \mathfrak{c}$ implies (2). $\mathfrak{b} = \mathfrak{d}$ implies (3). Only the obvious implications from these statements are known. Each axiom proving (6) also shows

(3). However, it is unknown whether (6) implies (5), (5) implies (4) or (4) implies (3). There are no known "consistent no" results about any of these.

References

[1] Kunen, K. On paracompactness of box products of compact spaces, Trans. Amer. Math. Soc. 240 (1987) p. 307-316
[2] Lawrence, L. B., Failure of normality in the box product of uncountably many real lines. Trans. Amer. Math. Soc. 348 (1996), 187–203.
[3] Roitman, J., More paracompact box products. Proc. Amer. Math. Soc. 74 (1979), no. 1, 171–176.
[4] Rudin, M.E, The box product of countably many compact matric spaces, Gen. Top. Appl. 2 (1972) 293-248
[5] Williams, S. W., Box Products. The Handbook for Set-theoretic Topology, North-Holland (1984) K. Kunen and J. Vaughan, eds., 169-200.
[6] Williams, S. and Yang, S., On the countable box products of compact ordinal spaces, Topology Proceedings 12 (1987), 159-172.

3.6. **Scott Williams.** Is there a notion of "dimension" for metric spaces satisfying:

(1) The dimension of Euclidean n-space is n.
(2) The dimension of the product of metric spaces (under the sup metric) is the sum of the dimensions of the factors.
(3) Dimension is non-increasing under distance non-increasing maps.
(4) The dimension is unchanged under dense subspaces.

Note: (3) says this definition is affected by metrics even though the topology is the same. For compact metric spaces, the Hausdorff dimension satisfies (1)–(3).

4. Geometric Topology and Geometric Group Theory

4.1. Greg Bell. Geometric group theory

1. Overview Geometric group theory studies a group (usually finitely generated) from the geometric point of view. For example, on could study the Cayley graph of a finitely generated group with respect to a finite generating set. Because different choices of generating sets give rise to different metric spaces, one puts an equivalence relation on two metric spaces, saying that they are the same if they are quasi-isometric. This equivalence gives rise to what is sometimes known as the large-scale or asymptotic approach to groups. Gromov began the study of so-called asymptotic invariants of infinite groups in [?]. Certain invariants are still the subject of much active research today. One

immediately sees that there is a strong interest in the interaction between large-scale dimension and group theory and there is a great deal of work involving non-positive curvature of groups. There are several notions of dimensions of groups: asymptotic dimension, cohomological dimension, Assouad-Nagata dimension, etc. It is known, [?], that the cohomological dimension of a group of type FP is no more than the asymptotic dimension. Also, Piotr Nowak has identified finitely generated groups with asymptotic dimension 2 whose Assouad-Nagata dimension is infinite. Indeed the relationships between these dimensions remain a mystery in general.

2. Questions about Geometric Group Theory In addition to the open questions listed by J. Dydak, one could ask: Question 1. Is the asymptotic dimension always bounded below by the cohomological dimension for groups? Question 2. Dan Margalit [?] has computed the cohomological dimension of the Torelli subgroup of Out(Fn). Is the asymptotic dimension of this group finite? If it is finite, what is it? We observe that this question is very closely related to the corresponding question for Mapping Class Groups. In particular, it is known that the Torelli subgroup of the mapping class group has finite asymptotic dimension when genus is less than 3. For higher genus this is unknown and is equivalent to the following question [?]: Question 3. Is the asymptotic dimension of the mapping class group of a surface with genus at least 3 finite? 1

3. A group is said to be exact if it admits a topologically amenable action on its Stone-Cech compactification, see [?]. It is also known that groups with finite asymptotic dimension are exact ([?]). A natural question, then, is the following: Question 4. Are the mapping class groups of surfaces with genus at least 3 exact? Are the corresponding Torelli groups exact? The non-positive curvature condition can be brought in at this point. It is known [?] that a hyperbolic group has finite asymptotic dimension (and is therefore exact). The corresponding question for non-positively curved groups (or CAT(0) groups) remains open. This remains one of the largest unsolved problems in this area of interaction between curvature and dimension: Question 5. Do CAT(0) groups have finite asymptotic dimension? In some sense, between CAT(0) groups and hyperbolic groups lie groups that are CAT(0) with isolated flats. As these are so-called relatively hyperbolic, these groups have finite asymptotic dimension by a result of Osin, [?]. Another large, well-studied class of groups for which the finiteness of asymptotic dimension remains unknown is automatic groups. Question 6. Do automatic groups have finite asymptotic dimension? Are automatic groups exact?

References

1. G. Bell and K. Fujiwara, The asymptotic dimension of a curve graph is finte, arXiv: math.GT/0509216, (2005).

2. M. Bestvina, K.-U. Bux and D. Margalit, Dimension of the Torelli group for Out(Fn), arXiv: math.GT/0603177 (2006).

3. A. Dranishnikov, Cohomological approach to asymptotic dimension, available on-line at http://arXiv.org/abs/math.MG/0608215, (2006).

4. M. Gromov, Asymptotic invariants of infinite groups, Geometric Group Theory, London Math. Soc. Lecture Note Ser. (G. Niblo and M. Roller, eds.), no. 182, 1993.

5. N. Higson and J. Roe, Amenable group actions and the Novikov conjecture, J. Reine Angew. Math. 519 (2000), 143153.

6. D. Osin, Asymptotic dimension of relatively hyperbolic groups, Int. Math. Res. Not. (2005), no. 35, 21432161.

4.2. **Michael Davis. The Eilenberg-Ganea problem:** Given a discrete group G, its *geometric dimension*, $gd(G)$, is the smallest dimension of a $K(G, 1)$ complex; its *cohomological dimension*, $cd(G)$, is the lenght of the shortest projective resolution of the trivial G-module. Obviously, $gd(G) \geq cd(G)$. The Eilenberg-Ganea Problem asks whether equality always holds. By work of Eilenberg-Ganea and Stallings and Swan, the only possibility for a counterexample would have $cd(G) = 2$ and $gd(G) = 3$. It is conjectured that counterexamples can be constructed using Coxeter groups. For example, let L be a two dimensional acyclic complex which is not simply connected. Let W be the right-angled Coxeter group with one generator of order 2 for each vertex of L and relations that two generators commute whenever they are connected by an edge. Let G be a torsion-free subgroup of finite index in W. Then $cd(G) = 2$. It seems plausible that $gd(G)$ is always 3.

4.3. **Alexander Drainisnikov. Aperiodic colorings of groups**

Problem: Does a finitely generated group Γ with $asdim(\Gamma) = n$ admit a coarse embedding into product of $n + 1$ binary trees.?

Definition. A *coloring of a set X by the set of colors F* is a map $\phi : X \to F$. We consider the product topology on the set of all colorings F^X of X where F is taken with the discrete topology. A coloring $\phi : \Gamma \to F$ of a discrete group Γ is called aperiodic if $\phi \neq \phi \circ g$ for all $g \in \Gamma, g \neq e$. A coloring $\phi : \Gamma \to F$ of a discrete group is called limit aperiodic if every coloring $\psi \in \overline{\phi\Gamma} \subset F^\Gamma$ is aperiordic.

Question: Does every group admit a limit aperiodic coloring by finitely many colors?

So far an affirmative answer is given for Coxeter groups and Gromov hyperbolic groups.

4.4. **J. Dydak. Future of asymptotic dimension theory.** The most interesting set of questions in asymptotic dimension theory deals with various characterizations of the three main dimensions; asymptotic dimension, Assouad-Nagata dimension, and asymptotic Assouad-Nagata dimension. As in cohomological dimension theory, where the geometrically defined covering dimension has, as algebraic counterpart, the integral dimension, there are three basic pairs of dimensions:

a. Asymptotic dimension and the dimension of Higson corona,
b. Asymptotic Assouad-Nagata dimension and the dimension of the sublinear Higson corona,
c. Assouad-Nagata dimension and the smallest n such that S^n is a Lipschitz extensor of the space.

In each case it is known that if the first dimension is finite, then both of them are equal. However, no example is known of the first dimension being infinite and the second dimension finite. In cohomological dimension it took 50 years to find a compact space with finite integral dimension and infinite covering dimension. Hopefully, based on experience gained, the time needed to untangle the differences between above pairs of dimensions will be shorter.

Asymptotic dimension theory of groups is fairly developed. The most interesting problem left open is if mapping class groups have finite asymptotic dimension. Also, in case it is finite it would be of interest to tie it to some geometrical property.

However, in case of Assouad-Nagata dimension of groups, not much is known. The most pressing issue is establishing its finiteness for basic classes of groups (nilpotent groups, polycyclic groups, mapping class groups). The second issue is to establish if Assouad-Nagata dimension and asymptotic dimension coincide for those classes of groups.

4.5. **Jennifer Schultens.** There a problem I would like to mention: *Can the connect sum of two unstabilized Heegaard splittings be stabilized?*

A handlebody is a 3-manifold with boundary that is a 3-dimensional fattening of a graph. A Heegaard splitting is a decomposition of a 3-manifold via a surface that cuts the 3-manifold into two handlebodies. Given a Heegaard splitting of a 3-manifold, one can add a trivial handle to the Heegaard splitting to obtain a new Heegaard splitting. This operation is called stabilization.

There are currently two independent parties claiming an affirmative answer to the above problem. One party is David Bachman, the other is Ruifeng Qiu. So far, neither argument has been verified.

4.6. **Bob Williams.** One of the favorite properties of geometric topologists is that of indecomposability; it comes up repeatedly in dynamics. For example, the inverse limit of the double cover of the circle, yields a dyadic solenoid, which is indedcomposable. So are 1 dimensional 'tiling spaces'. However, inverse limits of similar maps of higher dimensional tori are NOT indecomposable. Nor are 'tiling spaces' of dimension larger than 1. Is there a concept 'like' indecomposablity that captures this—in some ways very similar— structure?