MORE ZEROS OF THE DERIVATIVES OF THE RIEMANN ZETA FUNCTION ON THE LEFT HALF PLANE

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Abstract. We present the zeros of the derivatives, $\zeta^{(k)}(\sigma+it)$, of the Riemann zeta function for $k \le 28$ with $-10 < \sigma < \frac{1}{2}$ and $-10 < t < 10$. Our computations show an interesting behavior of the zeros of $\zeta^{(k)}$, namely they seem to lie on curves which are extensions of certain chains of zeros of $\zeta^{(k)}$ that were observed on the right half plane.

1. Introduction

Let $s \in \mathbb{C}$. We denote the real part of s by σ and the imaginary part of s by t. For $\sigma > 1$ the Riemann zeta function ζ can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

By analytic continuation, ζ may be extended to the whole complex plane, with the exception of the simple pole $s = 1$. This analytic continuation is characterized by the functional equation

$$\zeta(1-s) = 2\Gamma(s)\zeta(s)(2\pi)^{-s}\cos\frac{\pi s}{2}. \tag{2}$$

It follows directly from the functional equation (2) that $\zeta(-2j) = 0$ for all $j \in \mathbb{N}$. These zeros are called the real or trivial zeros of ζ. By the Riemann hypothesis, the remaining (non-trivial) zeros of ζ are of the form $\frac{1}{2} + it$.

In this paper we numerically investigate the distribution of zeros of the derivatives $\zeta^{(k)}$ of ζ on the left half plane. The results of our computations, that considerably expands the list of previously published zeros [11, 15], can be found in Table 1 and Table 2. For the rectangular region $-10 < \sigma < \frac{1}{2}$ and $|t| < 10$, Table 1 contains the number of zeros of $\zeta^{(k)}$, its real zeros, and its zeros with $0 < \sigma < \frac{1}{2}$. Table 2 contains non-real zeros with $\sigma < 0$ in that region. We find that some of the conjectured chains of zeros of the derivatives on the right half plane [9, 3] (see Figure 1) appear to continue to the left half plane which is illustrated in Figure 3.

We first recall results about the distribution of the zeros of $\zeta^{(k)}$ on the right half plane (Section 2) and the left half plane (Section 3). Section 4 contains a description of the methods we used to evaluate $\zeta^{(k)}$. It is followed by a discussion of the methods that we used to find the zeros of $\zeta^{(k)}$ in Section 5.

2. Zeros on the Right Half Plane

Assuming the Riemann Hypothesis, the non-real zeros of ζ are all on the critical line $\sigma = \frac{1}{2}$, while the non-real zeros of $\zeta^{(k)}$ appear to be distributed mostly to the right of the critical line with some outliers located to its left.

Zeros with $0 < \sigma < \frac{1}{2}$. Speiser related the Riemann Hypothesis to the distribution of zeros of the first derivative.

Theorem 1 (Speiser [10]). *The Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < \frac{1}{2}$.*

A simpler and more instructive proof of this result was given by Levinson and Montgomery [8]. They also proved, assuming the Riemann Hypothesis, that $\zeta^{(k)}(s)$ has at most a finite number of non-real zeros with $\sigma < \frac{1}{2}$, for $k \ge 2$.

k	# of zeros of $\zeta^{(k)}(\sigma+it)$ $-10<\sigma<0$			zeros of $\zeta^{(k)}(\sigma+it)$ $-10<\sigma<0$				$0<\sigma<1/2$
	$\|t\|<10$	$0<t<10$	$t=0$	$t=0$				$\|t\|<10$
0	4	0	4	-2	-4	-6	-8	
1	3	0	3	-2.7173	-4.9368	-7.0746		
2	5	1	3	-3.5958	-6.0290	-8.2786		
3	5	2	3	-4.7157	-7.2920	-9.6047		
4	6	2	2	-6.1265	-8.7016			
5	5	2	1	-7.7119				$0.2876 \pm 4.6944i$
6	7	2	3	-4.3284	-6.6083	-9.3445		
7	8	3	2	-5.6191	-8.4425			
8	7	3	1	-7.5186				$0.4183 \pm 5.4753i$
9	9	3	3	-4.7059	-6.5553	-9.3794		
10	10	4	2	-5.7309	-8.5500			
11	9	4	1	-7.7120				$0.4106 \pm 6.1502i$
12	11	4	3	-5.1849	-6.8533	-9.6751		
13	12	5	2	-6.1124	-8.9100			
14	11	5	1	-8.1400				$0.3447 \pm 6.7636i$
15	12	5	2	-5.6697	-7.3600			
16	14	6	2	-6.6469	-9.4393			
17	13	6	1	-8.7229				$0.2494 \pm 7.3344i$
18	14	6	2	-6.1556	-8.0019			
19	15	7	1	-7.3040				
20	15	7	1	-9.4151				0.1378 ± 7.8732
21	16	7	2	-6.6561	-8.7394			
22	17	8	1	-8.0675				
23	16	8	0					$0.0163 \pm 8.3861i$
24	18	8	2	-7.1929	-9.5491			$0.4681 \pm 8.7645i$
25	19	9	1	-8.9089				
26	20	9	2	-7.3618	-8.2504			
27	19	9	1	-7.8131				$0.3116 \pm 9.244i$
28	21	10	1	-9.8049				
29	22	10	2	-7.7492	-9.1919			
30	21	10	1	-8.6103				$0.1516 \pm 9.7083i$
31	22	11	0					
32	23	11	1	-8.2087				

TABLE 1. The number of zeros of $\zeta^{(k)}(\sigma+it)$ with $k \le 32$ in $-10<\sigma<0$, $|t|<10$, the number of complex conjugate pairs of non-real zeros, and the number of real zeros in this region. Furthermore, the real zeros in this region and the zeros in the strip $0<\sigma<\frac{1}{2}$, $|t|<10$ are given. The zeros are rounded to 4 decimal digits.

Theorem 2 (Yıldırım [15]). *The Riemann Hypothesis implies that ζ'' and ζ''' have no zeros in the strip $0 \le \sigma \le \frac{1}{2}$.*

The Riemann Hypothesis also implies that $\zeta^{(k)}$ for $k>0$ has only finitely many zeros in $0 \le \sigma \le \frac{1}{2}$ [8].

Our computations show that higher derivatives have zeros in this strip, see Table 1. Because of the distribution of the zeros of $\zeta^{(k)}$ in Figure 2, we expect that the zeros listed in the table are the only zeros of $\zeta^{(k)}$ for $k \le 32$.

Zeros with $\sigma > \frac{1}{2}$. The real parts of the zeros of $\zeta^{(k)}$ can be effectively bounded from above by absolute constants. For ζ' and ζ'' Skorokhdov [9] gives the bounds:

$$\zeta'(\sigma+it) \neq 0 \quad \text{for} \quad \sigma > 2.93938,$$
$$\zeta''(\sigma+it) \neq 0 \quad \text{for} \quad \sigma > 4.02853.$$

For $k \ge 3$ such general upper bounds were given by Spira [11] and later improved by Verma and Kaur [14]:

$$\zeta^{(k)}(\sigma+it) \neq 0 \quad \text{for} \quad \sigma > q_2 k + 2,$$

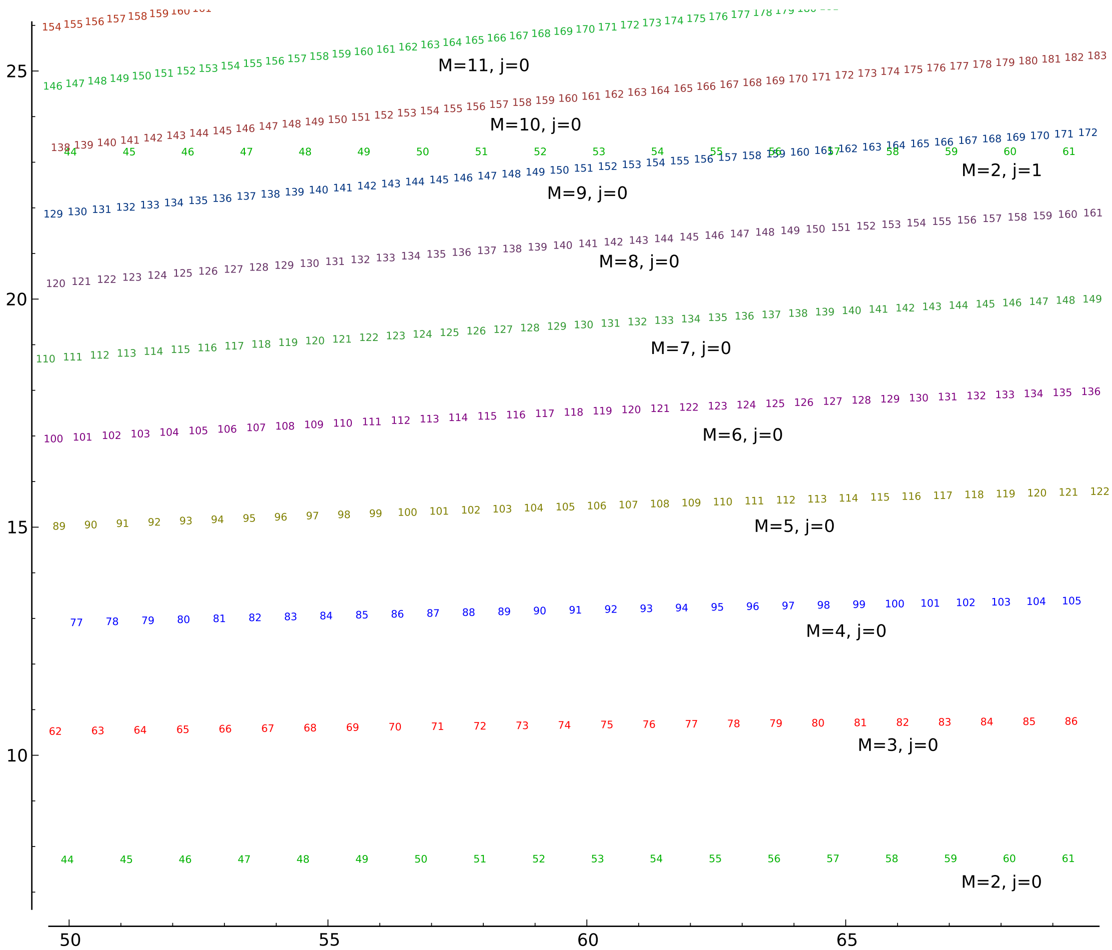

FIGURE 1. The zeros of $\zeta^{(k)}(\sigma+it)$ for $50<\sigma<70$, $0<t<26$, where k denotes a zero of $\zeta^{(k)}$. The conjectured chains of zeros are labeled by M and j (compare Theorem 3).

where q_2 is given by the formula

$$q_M = \frac{\log\left(\frac{\log M}{\log(M+1)}\right)}{\log\left(\frac{M}{M+1}\right)}.$$

Spira [11] computed zeros of the first and second derivative of $\zeta(s)$ for $0<t<100$ and noticed that they occur in pairs. Skorokhodov [9] went further in his computation and noticed that the zeros of derivatives of ζ seem to form chains, that is for each zero $z^{(k)}$ of $\zeta^{(k)}$ there seems to be a corresponding zero $z^{(k+1)}$ of $\zeta^{(k+1)}$. Indeed, for sufficiently large k the existence of these chains is a direct consequence of the following theorem.

Theorem 3 (Binder, Pauli, Saidak [3])**.** *Let $M\geq 2$ be an integer and let u be a solution of $1-\frac{1}{e^u 1}-\frac{1}{e^u}\left(1+\frac{1}{u}\right)\geq 0$, that is, $u\geq 1.1879\ldots$. If $k>\frac{u(2M+3)}{q_M-q_{M+1}}$ then for each $j\in\mathbb{Z}$ the rectangular region R, consisting of all $s=\sigma+it$ with*

$$q_M k-(M+1)u<\sigma<q_M k+(M+1)u \tag{3}$$

and

$$\frac{2\pi j}{\log(M+1)-\log(M)}<t<\frac{2\pi(j+1)}{\log(M+1)-\log(M)}, \tag{4}$$

contains exactly one zero of $\zeta^{(k)}$. This zero is simple.

So, given $M \geq 2$, $j \in \mathbb{Z}$ and $l > \frac{u(2M+3)}{q_M - q_{M+1}}$ for the zero of $\zeta^{(l)}$ in the region determined by (3) and (4) for $k = l$ there is a corresponding zero of $\zeta^{(l+1)}$ in the region determined by (3) and (4) for $k = l+1$. Figure 1 illustrates the phenomenon of the chains of zeros of derivatives of ζ. The zeros shown in the chains labeled $M = 2, j = 0$ and $M = 2, j = 1$ are in the rectangular regions from Theorem 3 and the zeros in the chain labeled $M = 3, j = 1$ are in the regions for $M = 3$ and $j = 1$ starting at the 77th derivative. The other chains are labeled by the parameters M and j of the regions into which higher derivatives in the chains eventually fall farther to the right.

3. Zeros on the Left Half Plane

It follows immediately from the functional equation (2) that $\zeta(s) = 0$ for $s = -2n$ where $n \in \mathbb{N}$. The zeros of the first derivative are exactly the zeros postulated by the theorem of Rolle.

Theorem 4 (Levinson and Montgomery [8]). *For $n \geq 2$ there is exactly one zero of ζ' in the interval $(-2n, -2n+2)$ and there are no other zeros of ζ' with $\sigma \leq 0$.*

Unlike on the right half plane, on the left there is no general (left) bound for the non-real zeros of $\zeta^{(k)}$. Spira showed:

Theorem 5 (Spira [12]). *For $k > 0$ there is an α_k so that $\zeta^{(k)}$ has only real zeros for $\sigma < \alpha_k$, and exactly one real zero in each open interval $(-1-2n, 1-2n)$ for $1 - 2n < \alpha_k$.*

The location of a zero of the second derivative on the left half plane shows up in [11]. For both $\zeta''(s)$ and $\zeta'''(s)$ Yıldırım [15] proved the existence of exactly one pair of conjugate non-trivial zeros with $\sigma < 0$ and gave their location.

Theorem 6 (Levinson and Montgomery [8]). *If $\zeta^{(k)}$ has only a finite number of non-real zeros in $\sigma < 0$ then $\zeta^{(k+1)}$ has the same property.*

Hence, the absolute value of the non-real zeros of $\zeta^{(k)}$ on the left half plane can be bounded. This can be done by iteratively generalizing Yıldırıms methods for the second and third derivatives to higher derivatives.

Table 2 contains all the zeros of $\zeta^{(k)}(\sigma + it)$ with $-10 < \sigma < 0$, $0 < |t| < 10$ for $2 \leq k \leq 29$. The patterns of the distribution of zeros in Figure 2 suggest that these are all the zeros for these derivatives on the left half plane.

4. Evaluating $\zeta^{(k)}$ on the left half plane

Methods for evaluating ζ and $\zeta^{(k)}$ include Euler-Maclaurin summation (see, for example [4]) or convergence acceleration for alternating sums [2]. Implementations for the evaluation of ζ can be found in various computer algebra systems. The Python library mpmath [6] contains functions for evaluating derivatives of Hurwitz zeta functions, and thus $\zeta^{(k)}$, on the right half plane using Euler-Maclaurin summation.

We considered two different approaches for evaluating $\zeta^{(k)}$ in the left half plane. Because of speed and ease of implementation we use Euler-Maclaurin summation rather than the derivatives of the functional equation (see [1] for formulas for these). Using Euler-Maclaurin summation we obtain for $\sigma = \Re(s) > 1$ that

$$\begin{aligned}(-1)^k \zeta^{(k)}(s) &= \sum_{n=2}^{\infty} \frac{\log^k(n)}{n^s} = \sum_{n=2}^{N-1} \frac{\log^k(n)}{n^s} + \sum_{n=N}^{\infty} \frac{\log^k(n)}{n^s} \\ &= \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \int_N^{\infty} \frac{\log^k(x)}{x^s} dx + \frac{1}{2} \frac{\log^k(N)}{N^s} + \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s} \bigg|_{x=N}^{\infty} + R_{2v} \\ &= \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \int_N^{\infty} \frac{\log^k(x)}{x^s} dx + \frac{1}{2} \frac{\log^k(N)}{N^s} - \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s} \bigg|_{x=N} + R_{2v},\end{aligned}$$

k	#	Zeros of $\zeta^{(k)}(\sigma+it)$ with $-10<\sigma<0$ and $0<\lvert t\rvert<10$			
2	1	$-0.3551 \pm 3.5908i$			
3	1	$-2.1101 \pm 2.5842i$			
4	2	$-0.8375 \pm 3.8477i$	$-3.2403 \pm 1.6896i$		
5	2	$-2.1841 \pm 3.0795i$	$-4.2739 \pm 0.6624i$		
6	2	$-1.2726 \pm 4.0742i$	$-3.1694 \pm 2.2894i$		
7	3	$-0.4133 \pm 4.8453i$	$-2.3934 \pm 3.4063i$	$-3.8750 \pm 1.4918i$	
8	3	$-1.6703 \pm 4.2784i$	$-3.2523 \pm 2.7170i$	$-4.5682 \pm 0.8112i$	
9	3	$-0.9672 \pm 4.9985i$	$-2.6410 \pm 3.6749i$	$-3.9459 \pm 2.0452i$	
10	4	$-0.2748 \pm 5.6133i$	$-2.0391 \pm 4.4684i$	$-3.4229 \pm 3.0609i$	$-4.5121 \pm 1.3321i$
11	4	$-1.4413 \pm 5.1493i$	$-2.9062 \pm 3.9132i$	$-4.0769 \pm 2.4384i$	$-5.0310 \pm 0.7641i$
12	4	$-0.8452 \pm 5.7473i$	$-2.3874 \pm 4.6486i$	$-3.6307 \pm 3.3459i$	$-4.6218 \pm 1.8307i$
13	5	$-0.2500 \pm 6.2811i$	$-1.8653 \pm 5.2971i$	$-3.1788 \pm 4.1283i$	$-4.2445 \pm 2.7740i$,
		$-5.1019 \pm 1.1817i$			
14	5	$-1.3402 \pm 5.8783i$	$-2.7202 \pm 4.8199i$	$-3.8543 \pm 3.5969i$	$-4.7812 \pm 2.1996i$,
		$-5.5404 \pm 0.6780i$			
15	5	$-0.8124 \pm 6.4056i$	$-2.2551 \pm 5.4415i$	$-3.4521 \pm 4.3265i$	$-4.4411 \pm 3.0614i$,
		$-5.2367 \pm 1.6383i$			
16	6	$-0.2827 \pm 6.8886i$	$-1.7845 \pm 6.0069i$	$-3.0400 \pm 4.9834i$	$-4.0887 \pm 3.8241i$,
		$-4.9528 \pm 2.5231i$	$-5.6490 \pm 1.0311i$		
17	6	$-1.3092 \pm 6.5262i$	$-2.6197 \pm 5.5821i$	$-3.7242 \pm 4.5121i$	$-4.6486 \pm 3.3161i$,
		$-5.4130 \pm 1.9836i$	$-6.0680 \pm 0.5743i$		
18	6	$-0.8299 \pm 7.0068i$	$-2.1924 \pm 6.1331i$	$-3.3491 \pm 5.1402i$	$-4.3279 \pm 4.0324i$,
		$-5.1468 \pm 2.8068i$	$-5.8098 \pm 1.4611i$		
19	7	$-0.3475 \pm 7.4543i$	$-1.7592 \pm 6.6440i$	$-2.9648 \pm 5.7192i$	$-3.9939 \pm 4.6871i$,
		$-4.8654 \pm 3.5483i$	$-5.5889 \pm 2.2963i$	$-6.1583 \pm 0.88585i$	
20	7	$-1.3211 \pm 7.1206i$	$-2.5729 \pm 6.2569i$	$-3.6489 \pm 5.2913i$	$-4.5694 \pm 4.2268i$,
		$-5.3472 \pm 3.0608i$	$-5.9945 \pm 1.7820i$	$-6.6140 \pm 0.43943i$	
21	7	$-0.8787 \pm 7.5677i$	$-2.1744 \pm 6.7594i$	$-3.2944 \pm 5.8530i$	$-4.2605 \pm 4.8536i$,
		$-5.0870 \pm 3.7617i$	$-5.7837 \pm 2.5734i$	$-6.3545 \pm 1.2934i$	
22	8	$-0.4328 \pm 7.9887i$	$-1.7703 \pm 7.2313i$	$-2.9319 \pm 6.3785i$	$-3.9406 \pm 5.4371i$,
		$-4.8118 \pm 4.4095i$	$-5.5554 \pm 3.2943i$	$-6.1750 \pm 2.0870i$	$-6.6413 \pm 0.7581i$
23	8	$-1.3613 \pm 7.6765i$	$-2.5625 \pm 6.8727i$	$-3.6113 \pm 5.9836i$	$-4.5240 \pm 5.0128i$,
		$-5.3115 \pm 3.9611i$	$-5.9806 \pm 2.8250i$	$-6.5366 \pm 1.5912i$	$-7.1892 \pm 0.1700i$
24	8	$-0.9481 \pm 8.0980i$	$-2.1871 \pm 7.3395i$	$-3.2737 \pm 6.4980i$	$-4.2254 \pm 5.5784i$,
		$-5.0539 \pm 4.5827i$	$-5.7671 \pm 3.5097i$	$-6.3712 \pm 2.3553i$	$-6.8798 \pm 1.1259i$
25	9	$-0.5313 \pm 8.4984i$	$-1.8064 \pm 7.7820i$	$-2.9291 \pm 6.9843i$	$-3.9174 \pm 6.1112i$,
		$-4.7841 \pm 5.1658i$	$-5.5378 \pm 4.1485i$	$-6.1844 \pm 3.0574i$	$-6.7253 \pm 1.8906i$,
		$-7.1206 \pm 0.6504i$			
26	9	$-0.1113 \pm 8.8798i$	$-1.4211 \pm 8.2028i$	$-2.5782 \pm 7.4458i$	$-3.6013 \pm 6.6153i$,
		$-4.5038 \pm 5.7155i$	$-5.2952 \pm 4.7478i$	$-5.9817 \pm 3.7117i$	$-6.5664 \pm 2.6042i$,
		$-7.0463 \pm 1.4126i$			
27	9	$-1.0318 \pm 8.6041i$	$-2.2218 \pm 7.8850i$	$-3.2780 \pm 7.0941i$	$-4.2144 \pm 6.2361i$,
		$-5.0410 \pm 5.3132i$	$-5.7647 \pm 4.3261i$	$-6.3901 \pm 3.2731i$	$-6.9206 \pm 2.1489i$,
		$-7.3814 \pm 0.9448i$			
28	10	$-0.6389 \pm 8.9878i$	$-1.8606 \pm 8.3044i$	$-2.9484 \pm 7.5503i$	$-3.9169 \pm 6.7308i$,
		$-4.7767 \pm 5.8489i$	$-5.5353 \pm 4.9061i$	$-6.1978 \pm 3.9018i$	$-6.7680 \pm 2.8338i$,
		$-7.2490 \pm 1.7019i$	$-7.6182 \pm 0.5486i$		
29	10	$-0.2428 \pm 9.3554i$	$-1.4951 \pm 8.7056i$	$-2.6132 \pm 7.9860i$	$-3.6122 \pm 7.2024i$,
		$-4.5034 \pm 6.3583i$	$-5.2947 \pm 5.4558i$	$-5.9918 \pm 4.4954i$	$-6.5986 \pm 3.4759i$,
		$-7.1165 \pm 2.3954i$	$-7.5353 \pm 1.2495i$		
30	10	$-1.1257 \pm 9.0905i$	$-2.2729 \pm 8.4034i$	$-3.3013 \pm 7.6533i$	$-4.2222 \pm 6.8443i$,
		$-5.0444 \pm 5.9789i$	$-5.7739 \pm 5.0583i$	$-6.4149 \pm 4.0822i$	$-6.9700 \pm 3.0489i$,
		$-7.4393 \pm 1.9531i$	$-7.8300 \pm 0.7596i$		
31	11	$-0.7529 \pm 9.4602i$	$-1.9282 \pm 8.8039i$	$-2.9846 \pm 8.0854i$	$-3.9340 \pm 7.3091i$,
		$-4.7854 \pm 6.4781i$	$-5.5454 \pm 5.5941i$	$-6.2186 \pm 4.6575i$	$-6.8081 \pm 3.6673i$,
		$-7.3161 \pm 2.6210i$	$-7.7489 \pm 1.5152i$	$-8.1557 \pm 0.4150i$	
32	11	$-0.3770 \pm 9.8161i$	$-1.5795 \pm 9.1891i$	$-2.6629 \pm 8.5003i$	$-3.6395 \pm 7.7548i$,
		$-4.5188 \pm 6.9560i$	$-5.3075 \pm 6.1058i$	$-6.0109 \pm 5.2053i$	$-6.6324 \pm 4.2542i$,
		$-7.1745 \pm 3.2514i$	$-7.6387 \pm 2.1955i$	$-8.0192 \pm 1.0955i$	

TABLE 2. All zeros of $\zeta^{(k)}(\sigma+it)$ with $k \le 32$ in $-10<\sigma<0$, $0<\lvert t\rvert<10$. The column # contains the number of conjugate pairs of zeros. All zeros listed are simple and rounded to 4 decimal digits.

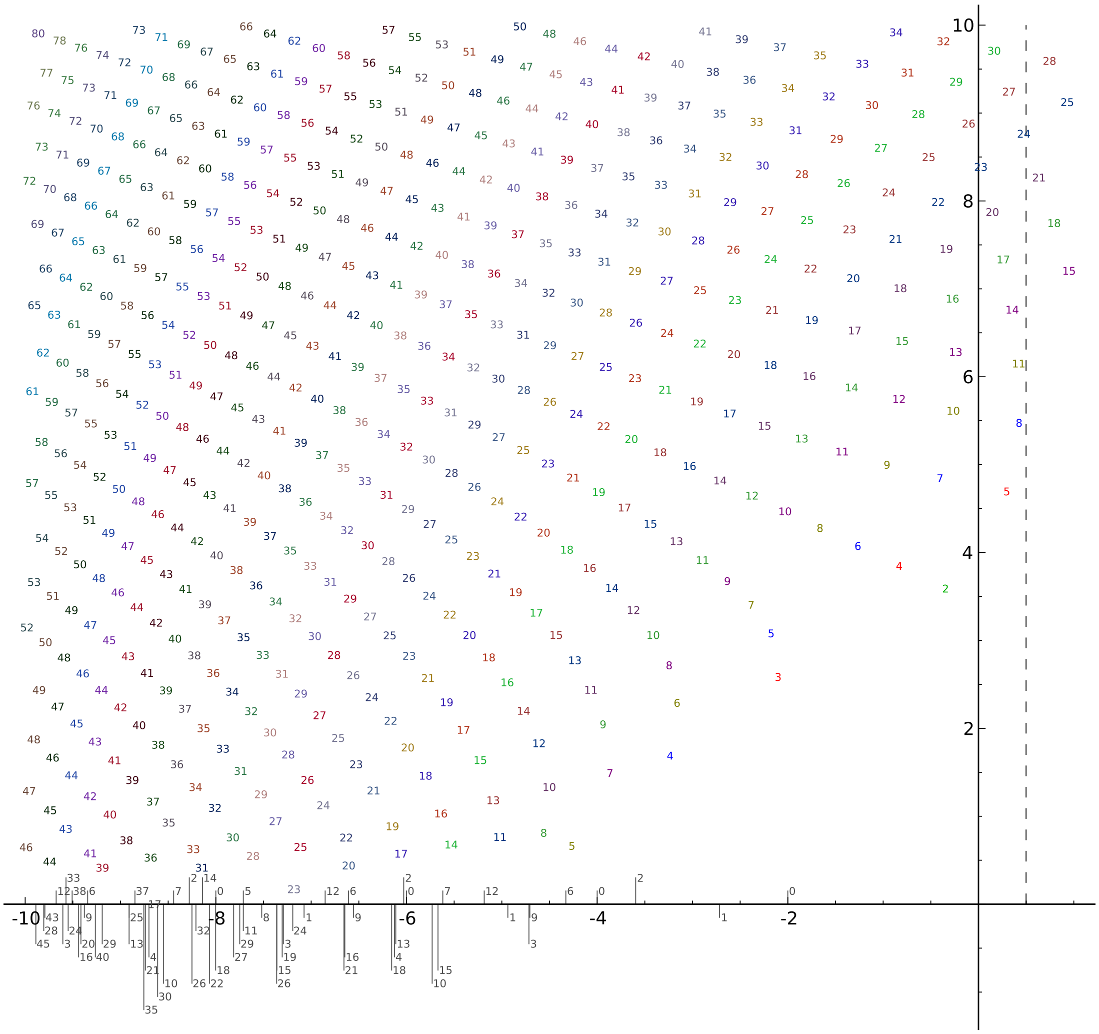

FIGURE 2. The zeros of $\zeta(\sigma+it)$ and its derivatives $\zeta^{(k)}(\sigma+it)$ for $k \leq 80$ in $-10 < \sigma < 1$, $0 < t < 10$, where 0 denotes a zero of ζ and k denotes a zero of $\zeta^{(k)}$. All zeros shown are simple.

where $N \in \mathbb{N}^{>2}$, $v \in \mathbb{N}^{>2}$, and R_{2v} is the error term. Repeated integration by parts yields:

$$\int_N^\infty \frac{\log^k(x)}{x^s}dx = \frac{\log^k(N)}{(s-1)N^{s-1}} \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{\log^{-r}(N)}{(s-1)^r}$$

Thus,

$$\begin{aligned} \zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{N-1} \frac{\log^k(s)}{n^s} + \frac{\log^k(N)}{(s-1)N^{s-1}} \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{\log^{-r}(N)}{(s-1)^r} + \frac{1}{2}\frac{\log^k(N)}{N^s} \\ - \sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} \frac{d^{2j-1}}{dx^{2j-1}} \frac{\log^k(x)}{x^s}\bigg|_{x=N} + R_{2v}, \end{aligned} \tag{5}$$

The error term R_{2v} is given by

$$R_{2v} = \frac{1}{(2v)!}\int_N^\infty \hat{B}_{2v}(x) f^{(2v)}(x)dx,$$

with $f(x) = \frac{\log^k(x)}{x^s}$ as discussed in [4]. We use the non-central Stirling numbers of the first kind (see [5]), to represent the derivatives of f. The non-central Stirling numbers of the first kind $S(r, i, s)$ satisfy the recurrence

$$\begin{aligned}
&S(1, 0, s) = -s \\
&S(1, 1, s) = 1 \\
&S(r + 1, 0, s) = (-s - r)S(r, 0, s) \\
&S(r + 1, i, s) = (-s - r)S(r, i, s) + S(r, i - 1, s) \text{ for } 1 \leq i \leq r \\
&S(r + 1, r + 1, s) = S(r, r, s)
\end{aligned}$$

With these the derivatives of f can be written as

$$f^{(r)}(x) = x^{-s-r} \sum_{i=0}^{r} S(r, i, s)(k)_i \log^{k-i}(x)$$

where $(k)_i$ denotes the i-th falling factorial of k [5].

We now bound the error term, R_{2v}. Observe that

$$\begin{aligned}
|R_{2v}| &= \left| \frac{1}{(2v)!} \int_N^\infty \hat{B}_{2v}(x) f^{(2v)}(x) dx \right| && (6) \\
&\leq \frac{|B_{2v}|}{(2v)!} \int_N^\infty |f^{(2v)}(x)| dx && (7) \\
&= \frac{|B_{2v}|}{(2v)!} \int_N^\infty \left| x^{-s-2v} \sum_{i=0}^{2v} S(2v, i, s)(k)_i \log^{k-i}(x) \right| dx && (8) \\
&\leq \frac{|B_{2v}|}{(2v)!} \sum_{i=0}^{2v} \int_N^\infty \left| S(2v, i, s)(k)_i \frac{\log^{k-i}(x)}{x^{s+2v}} \right| dx && (9) \\
&= \frac{|B_{2v}|}{(2v)!} \sum_{i=0}^{2v} |S(2v, i, s)|(k)_i \int_N^\infty \frac{\log^{k-i}(x)}{x^{\sigma+2v}} dx && (10) \\
&\leq \frac{|B_{2v}|}{(2v)!} \sum_{i=0}^{2v} |S(2v, i, s)|(k)_i \left(\int_N^\infty \frac{\log^{k}(x)}{x^{\sigma+2v}} dx \right) && (11)
\end{aligned}$$

The error term R_{2v} converges for $\sigma + 2v > 1$ and $N \in \mathbb{N}^{>2}$, thus (5) can be used to evaluate $\zeta^{(k)}$ for $\sigma > 1 - 2v$. Since we are evaluating $\zeta^{(k)}$ on a bounded region with $|\sigma| \leq 10$ the error can be bounded by (11) on the entire region. We set $v = 101$, which yields $\sigma + 2v > 1$ in the region and gives a good balance of the values for v and N. To determine the value N should take, we evaluate the bound given above for $N = 200, 300, \ldots$ until the error is as small as desired. For example, if $s = -10 + 10i$, $k = 100$, $v = 101$, and $N = 200$ then $|R_{2v}| < 1.769892 \cdot 10^{-100}$. If $N = 1500$ then $|R_{2v}| < 1.245704 \cdot 10^{-253}$.

5. Finding Zeros

We found the zeros on the left half plane by following the chains of zeros of derivatives of ζ from the right half plane (see Figures 1 and 3). For given $M \geq 2$, $j \in \mathbb{Z}$, and sufficiently large k the center

$$s = q_M k + \frac{2\pi(j + 0.5)}{\log(M + 1) - \log(M)}$$

of the rectangular region from Theorem 3 is a good approximation to the zero in this region which we improved using Newtons method.

Now assume that we know a zero $z_M^{(k)}$ of $\zeta^{(k)}$ and a zero $z_M^{(k+1)}$ of $\zeta^{(k+1)}$ in the chain given by some M and j. We used

$$s = z_M^{(k)} - \left(z_M^{(k+1)} - z_M^{(k)} \right)$$

as a first approximation for the zero of $\zeta^{(k-1)}$ in that chain, which again was improved with Newtons method.

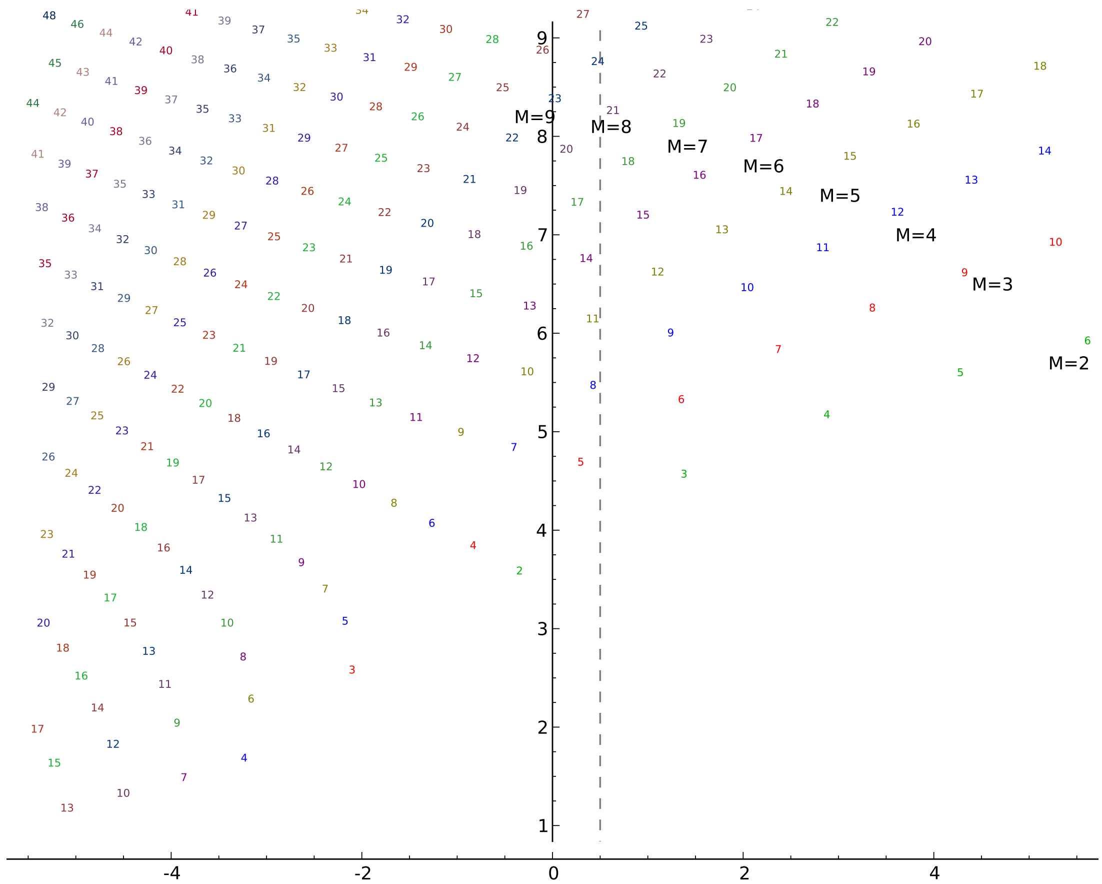

FIGURE 3. Zeros of $\zeta^{(k)}(\sigma + it)$. The zeros of $\zeta^{(k)}$ are at the center of the numbers k. The first five chains of zeros that we followed from the right to the left half plane are labeled $M = 2, \ldots, M = 6$ (see Section 2).

We assured that we had found all zeros of $\zeta^{(k)}$ with $0 < k \leq 61$ in $-10 < \sigma < \frac{1}{2}$, $|t| < 10$ by counting the zeros using contour integration. The only pole of $\zeta^{(k)}$ is at one and thus outside our region of interest. So for any simple closed contour C in $-10 < \sigma < \frac{1}{2}$, $|t| < 10$, by the argument principle, the number of zeros of $\zeta^{(k)}$ inside C is

$$n = \frac{1}{2\pi i} \int_C \left(\frac{\zeta^{(k+1)}}{\zeta^{(k)}} \right) (s) \, ds.$$

For $0 < k \leq 61$ we counted the zeros of $\zeta^{(k)}$ by integrating along the border of the rectangular region $-10 < \sigma < \frac{1}{2}$, $|t| < 10$. We also integrated along the sides of a square region with side length 10^{-6} centered around each approximation z of the zeros to make sure that this region contained exactly one zero.

All computations and plotting were conducted with the computer algebra system Sage [13]. We evaluated $\zeta^{(k)}$ with our implementation of the method described in Section 4 which was verified, on the right half plane, with the Hurwitz zeta function in mpmath [6] and our implementation of $\zeta^{(k)}$ based on convergence acceleration for alternating series. For the integration we used the numerical integration function of Sage which calls the GNU Scientific Library [7] using an adaptive Gauss-Kronrod rule.

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