ON FRACTIONAL STIELTJES CONSTANTS

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Abstract. We discuss the non-integral generalized Stieltjes constants $\gamma_\alpha(a)$ arising naturally from the Laurent series expansions of the fractional derivatives of the Hurwitz zeta functions $\zeta^{(\alpha)}(s,a)$, and we prove that if one defines $h_a(s) := \zeta(s,a) - 1/(s-1) - 1/a^s$ and $C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$, then

$$C_\alpha(a) \;=\; (-1)^{-\alpha} h_a^{(\alpha)}(1),$$

for all real $\alpha \geq 0$, where $h^{(\alpha)}(x)$ denotes the α-th Grünwald-Letnikov fractional derivative of the function h at x. This result confirms the conjecture of Kreminski [8], originally stated in terms of the Weyl fractional derivatives.

1. Introduction

The Hurwitz zeta function is defined, for $\Re(s) > 1$ and $0 < a \leq 1$, as $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. It can be extended to a meromorphic function with a simple pole at $s = 1$ with residue 1 (see [2], [4]). Moreover, the function has a Laurent series expansion about $s = 1$, given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)(s-1)^n}{n!}, \tag{1}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants. The original Stieltjes constants were defined in 1885 (see [10]), but are themselves a generalization of Euler's constant γ:

$$\gamma = \gamma_0(1) = \lim_{m\to\infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \log m \right) = 0.57721\ 56649\ \cdots$$

In 1972, Berndt [3] showed that for the generalized Stieltjes constants in (1) we have:

$$\gamma_k(a) = \lim_{m\to\infty} \left\{ \sum_{n=0}^{m} \frac{\log^k(n+a)}{n+a} - \frac{\log^{k+1}(m+a)}{k+1} \right\}. \tag{2}$$

Furthermore, it was established by Williams and Zhang [11] that

$$\gamma_k(a) = \sum_{r=0}^{m} \frac{\log^k(r+a)}{r+a} - \frac{\log^{k+1}(m+a)}{k+1} - \frac{\log^k(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f_k'(x)dx, \tag{3}$$

where $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

More recently, Kreminski [8] has given a generalization of $\gamma_r(a)$ to $r \in \mathbb{R}^{>0}$, the so-called *fractional Stieltjes constants*. He computed $C_r(a) = \gamma_r(a) - \frac{\log^r(a)}{a}$ and conjectured that this is equal to $(-1)^r$ times the r-th Weyl fractional derivative of $\zeta(s,a) - 1/(s-1) - 1/a^s$ at $s = 1$.

The main aim of our paper is to employ Grünwald-Letnikov fractional derivatives to prove this conjecture of Kreminski. In the process we generalize the results from Berndt [3] and Williams & Zhang [11] to the fractional case.

2. Fractional Derivatives

We begin by giving a brief summary of the most useful basic properties of generalized derivatives. Fractional derivative operators are generalizations of the familiar differentiation operator D^n to arbitrary (integer, rational, or complex) values of n.

For $N \in \mathbb{N}$, let $\Delta_h^N f(z) = (-1)^N \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(z+kh)$ be the finite difference of f. Then we have (see [9], for example): $f^{(n)}(z) = \lim_{h\to 0} \frac{\Delta_h^n f(z)}{h^n}$ for all $n \in \mathbb{N}$; and this can be naturally extended to the fractional case (cf. [5]) via

$$\Delta_h^\alpha f(z) = (-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. For any $\alpha \in \mathbb{C}$, the so-called reverse α^{th} Grünwald-Letnikov derivative of a function $f(z)$ is now defined as (see [7]):

$$D_z^\alpha [f(z)] = \lim_{h\to 0^+} \frac{\Delta_h^\alpha f(z)}{h^\alpha} = \lim_{h\to 0^+} \frac{(-1)^\alpha \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha} \tag{4}$$

whenever the limit exists. Thus defined, $D_z^\alpha[f(z)]$ coincide with the standard derivatives for all $\alpha \in \mathbb{N}$. Also, they are analytic functions of α and z (as long as $f(z)$ is analytic) and satisfy: $D_z^0[f(z)] = f(z)$ and $D_z^\alpha \left[D_z^\beta [f(z)]\right] = D_z^{\alpha+\beta} [f(z)]$.

Although the Grünwald-Letnikov derivative is defined for all $\alpha \in \mathbb{C}$, we only consider $\alpha \in \mathbb{R}$ with $\alpha \geq 0$ in this paper. The following two useful results can be found in [9].

Lemma 2.1. *Let $\alpha \in \mathbb{R}$, $\beta < 0$, and $z \in \mathbb{C}$ with $\Re(z) > 1$. Then,*

$$D_z^\alpha \left[(z-1)^\beta\right] = \frac{(-1)^\alpha \Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-1)^{\beta-\alpha}.$$

Lemma 2.2. *Let $\alpha \geq 0$, $a > 0$, and $z \in \mathbb{C}$. Then $D_z^\alpha [e^{-az}] = (-1)^\alpha a^\alpha e^{-az}$.*

In particular for the Hurwitz zeta function we obtain:

Corollary 2.3. *Let $0 < a \leq 1$, and $s \in \mathbb{C}$ with $\Re(s) > 1$. The Grünwald-Letnikov fractional derivative of order $\alpha \geq 0$ with respect to s of $\zeta(s,a) - 1/a^s$ is*

$$D_s^\alpha [\zeta(s,a) - 1/a^s] = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha (n+a)}{(n+a)^s}. \tag{5}$$

3. Fractional Stieltjes Constants

Let $\alpha \in \mathbb{R}$, $\alpha > 0$ and $0 < a \le 1$. We define the fractional Stieltjes constants to be the coefficients of the expansion

$$S^{\alpha}(s,a) := (-1)^{-\alpha} D_s^{\alpha}[\zeta(s,a) - 1/a^s] = \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+a)}{(n+a)^s}$$

$$= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n, \text{ where } s \neq 1. \tag{6}$$

As a generalization of (3) we obtain:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \le 1$ and $m \in \mathbb{N}$. We have*

$$\gamma_{\alpha}(a) = \sum_{r=0}^{m} \frac{\log^{\alpha}(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^{\alpha}(m+a)}{2(m+a)} + \int_m^{\infty} P_1(x) f_{\alpha}'(x)dx, \tag{7}$$

where $f_{\alpha}(x) = \frac{\log^{\alpha}(x+a)}{x+a}$ and $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

Letting $m \to \infty$ yields, for all $\alpha > 0$ and $0 < a \le 1$, a natural generalization of (2):

$$C_{\alpha}(a) := \gamma_{\alpha}(a) - \frac{\log^{\alpha}(a)}{a} = \lim_{m\to\infty} \left\{ \sum_{r=1}^{m} \frac{\log^{\alpha}(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} \right\}.$$

which Kreminski **??** uses to define $\gamma_{\alpha}(a)$ for $\alpha \in \mathbb{R}$.

Proof. We use the following form of the Euler-Maclaurin summation formula

$$\sum_{k=m}^{n} g(k) = \int_m^n g(x)dx + \sum_{k=1}^{v} \frac{(-1)^k B_k}{k!} g^{(k+1)}(x)\bigg|_m^n + (-1)^{v+1} \int_m^n P_v(x) g^{(v)}(x)dx, \tag{8}$$

where $v \in \mathbb{N}$, $g(x) \in C^v[m,n]$, and $P_k(x)$ is the k^{th} periodic Bernoulli polynomial

$$P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}.$$

We take $v = 1$ in (8) and choose $g(x) = \frac{\log^{\alpha}(x+a)}{(x+a)^s}$, for $Re(s) > 1$. Letting $n \to \infty$, we obtain

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{\log^{\alpha}(r+a)}{(r+a)^s} &= \sum_{r=0}^{m-1} \frac{\log^{\alpha}(r+a)}{(r+a)^s} + \int_m^{\infty} \frac{\log^{\alpha}(x+a)}{(x+a)^s}dx + \frac{\log^{\alpha}(m+a)}{2(m+a)^s} + \int_m^{\infty} P_1(x)g'(x)dx \\
&= \sum_{r=0}^{m} \frac{\log^{\alpha}(r+a)}{(r+a)^s} + \int_m^{\infty} \frac{\log^{\alpha}(x+a)}{(x+a)^s}dx - \frac{\log^{\alpha}(m+a)}{2(m+a)^s} + \int_m^{\infty} P_1(x)g'(x)dx \\
&=: A(s) + B(s) - D(s) + G(s).
\end{aligned}$$

For the first term $A(s)$ we have:

$$\begin{aligned}A(s) &= \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{(r+a)^s} = \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} e^{-(s-1)\log(r+a)} \\ &= \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} \sum_{n=0}^{\infty} \frac{(-1)^n \log^n(r+a)}{n!}(s-1)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n}{n!} \sum_{r=0}^{m} \frac{\log^{\alpha+n}(r+a)}{(r+a)}.\end{aligned}$$

Now, since $\alpha \geq 0$, $m \in \mathbb{N}$, and $0 < a \leq 1$, for all $s \in \mathbb{C}$ with $\Re(s) > 1$, the second term $B(s)$ can be written in terms of the Upper Incomplete Gamma function $\Gamma(\alpha, s)$ (compare with [6, p. 346] and [1, 6.5.3]) as follows:

$$\begin{aligned}B(s) &= \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx = \frac{\Gamma(\alpha+1, (s-1)\log(m+a))}{(s-1)^{\alpha+1}} \\ &= \frac{1}{(s-1)^{\alpha+1}} \left[\Gamma(\alpha+1) - (s-1)^{\alpha+1} \log^{\alpha+1}(m+a) \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n \log^n(m+a)}{(\alpha+1+n)n!} \right] \\ &= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \log^{\alpha+1}(m+a) \sum_{n=0}^{\infty} \frac{(-1)^n (s-1)^n \log^n(m+a)}{(\alpha+1+n)n!} \\ &= \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \sum_{n=0}^{\infty} \left(\frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} \right) \frac{(-1)^n (s-1)^n}{n!}.\end{aligned}$$

For the third term $D(s)$, we write:

$$\begin{aligned}D(s) &= \frac{\log^\alpha(m+a)}{2(m+a)^s} = \frac{\log^\alpha(m+a)}{2(m+a)} e^{-(s-1)\log(m+a)} \\ &= \frac{\log^\alpha(m+a)}{2(m+a)} \sum_{n=0}^{\infty} \frac{(-1)^n \log^n(m+a)(s-1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{\log^{\alpha+n}(m+a)}{2(m+a)} \right) \frac{(-1)^n (s-1)^n}{n!}.\end{aligned}$$

If we now define

$$E_{\alpha,m}(n) := \sum_{r=0}^{m} \frac{\log^{\alpha+n}(r+a)}{r+a} - \frac{\log^{\alpha+n+1}(m+a)}{\alpha+n+1} - \frac{\log^{\alpha+n}(m+a)}{2(m+a)},$$

then combining the above expressions for $A(s)$, $B(s)$ and $D(s)$ we get:

$$\begin{aligned}\sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{(r+a)^s} + \int_m^\infty \frac{\log^\alpha(x+a)}{(x+a)^s} dx - \frac{\log^\alpha(m+a)}{2(m+a)^s} \\ = \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} E_{\alpha,m}(n) \frac{(-1)^n (s-1)^n}{n!}.\end{aligned}$$

For the last term we have:

$$G(s) = \int_m^\infty P_1(x) g'(x) dx = \int_m^\infty P_1(x) \left[-s \frac{\log^\alpha(x+a)}{(x+a)^{s+1}} + \alpha \frac{\log^{\alpha-1}(x+a)}{(x+a)^{s+1}} \right] dx.$$

From the definition of the fractional Stieltjes constants we have:

$$\sum_{n=0}^\infty E_{\alpha,m}(n) \frac{(-1)^{\alpha+n}(s-1)^n}{n!} + G(s) = \sum_{n=0}^\infty \frac{(-1)^{\alpha+n}\gamma_{\alpha+n}(s-1)^n}{n!}.$$

Taking successive derivatives with respect to s, of both sides, and then evaluating them at $s=1$, we see that for all $n \in \mathbb{N} \cup \{0\}$,

$$\gamma_{\alpha+n}(a) = E_{\alpha,m}(n) + G^{(n)}(1). \tag{9}$$

Setting $n=0$ in (9) and noting that $G(1) = f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, we obtain

$$\begin{aligned}\gamma_\alpha(a) &= E_{\alpha,m}(0) + G(1) \\ &= \sum_{r=0}^m \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f'_\alpha(x) dx,\end{aligned}$$

which proves the result. □

Corollary 3.2. *As $\alpha \to 0^+$, $\gamma_\alpha \to \gamma - 1$ where $\gamma = \gamma_0(1)$ is Euler's constant.*

Proof. Observe that, with $a = 1$, the left-hand sum in (6) becomes $\sum_{n=0}^\infty \frac{\log^\alpha(n+1)}{(n+1)^s}$, which as $\alpha \to 0^+$ converges to

$$\sum_{n=1}^\infty \frac{1}{(n+1)^s} = \sum_{n=2}^\infty \frac{1}{n^s} = \zeta(s) - 1.$$

From the Laurent series expansion of $\zeta(s)$ about $s=1$, we have

$$\zeta(s) - 1 = \frac{1}{s-1} + \gamma - 1 + \sum_{n=2}^\infty \frac{(-1)^n \gamma_n(1) \cdot (s-1)^n}{n!}. \tag{10}$$

Hence, in order to maintain equality in (6), the right hand side of (6) must approach $\zeta(s)-1$ as $\gamma \to 0^+$. From observation, this occurs if and only if $\gamma_\alpha(1) \to \gamma - 1$ as $\alpha \to 0^+$. □

It follows that $\gamma_\alpha(1)$, as a function of α, is discontinuous at $\alpha = 0$.

4. Kreminski's Conjecture

Now we are ready to prove our main result, namely [8, Conjecture (IIIa)]:

Theorem 4.1. *Let $h_a(s) = \zeta(s,a) - \frac{1}{s-1} - \frac{1}{a^s}$ and let $h_a^{(\alpha)}(s) = D_s^\alpha [h_a(s)]$ be the α-th Grünwald-Letnikov fractional derivative of h_a. Then*

$$C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = (-1)^{-\alpha} h_a^{(\alpha)}(1).$$

Kreminski's [8] original statement of the conjecture is slightly different than ours, due to his use of the Weyl fractional derivative W_s^α. For $0 < \alpha < 1$ the Weyl fractional derivatives of the relevant functions are

$$W_s^\alpha \left[\frac{1}{s-1} \right] = \frac{(-1)^{-\alpha} \alpha\pi \csc(\alpha\pi)}{\Gamma(1-\alpha)(s-1)^{\alpha+1}} = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}}$$

and

$$W_s^\alpha \left[\zeta(s,a) - 1/a^s \right] = (-1)^{-\alpha} \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s}.$$

These differ by a factor $(-1)^{2\alpha}$ from the Grünwald-Letnikov fractional of the same functions (Lemma 2.1 and Corollary 2.3), just as our statement of Kreminski's conjecture differs from the original by the same factor.

Proof. We have

$$h_a^{(\alpha)}(s) = D_s^\alpha \left[h_a(s) \right] = D_s^\alpha \left[\zeta(s,a) - \frac{1}{a^s} \right] - D_s^\alpha \left[\frac{1}{s-1} \right].$$

With Lemma 2.1 and Corollary 2.3 we obtain

$$h_a^{(\alpha)}(s) = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{(-1)^\alpha \Gamma(\alpha+1)}{(s-1)^{\alpha+1}},$$

or equivalently

$$\begin{aligned}
(-1)^{-\alpha} h_a^{(\alpha)}(s) &= \sum_{n=1}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} \\
&= \sum_{n=0}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} - \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} - \frac{\log^\alpha(a)}{a} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!} (s-1)^n - \frac{\log^\alpha(a)}{a}.
\end{aligned}$$

Evaluating $h_a^{(\alpha)}(s)$ at the point $s = 1$, finishes the proof:

$$(-1)^{-\alpha} h_a^{(\alpha)}(1) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a} = C_\alpha(a).$$

□

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