A BOUND FOR FRACTIONAL STIELTJES CONSTANTS

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Abstract. We discuss methods of evaluation of non-integral generalized Stieltjes constants $\gamma_\alpha(a)$, arising naturally from the Laurent series expansions of the fractional derivatives of the Hurwitz zeta functions $\zeta^{(\alpha)}(s,a)$. We give upper bounds for $C_\alpha(a) = \gamma_\alpha(a) - \log^\alpha(a)/a$ for $1 < \alpha$ and observe that this bound improves on previously known bounds for (integral) generalized Stieltjes constants.

1. Introduction

The Hurwitz zeta function is defined, for $\Re(s) > 1$ and $0 < a \le 1$, as $\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. It can be extended to a meromorphic function with a simple pole at $s = 1$ with residue 1 (see [2], [4]). Moreover, the function has a Laurent series expansion about $s = 1$, given by

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)(s-1)^n}{n!}, \tag{1}$$

where $\gamma_n(a)$ are the generalized Stieltjes constants.

Using fractional derivatives the discrete value n in $\gamma_n(a)$ can be considered as a continuous value. Kreminski [11] has given a generalization of $\gamma_\alpha(a)$ to $\alpha \in \mathbb{R}^{>0}$ called *fractional Stieltjes constants* These can be defined as the coefficients of the Laurent expansion of the α-th Grünwald-Letnikov fractional derivative [8] of $\zeta(s,a) - 1/a^s$ for $s \neq 1$ [6]:

$$\begin{aligned} D_s^\alpha\left[\zeta(s,a) - 1/a^s\right] &= (-1)^{-\alpha} \sum_{n=0}^{\infty} \frac{\log^\alpha(n+a)}{(n+a)^s} \\ &= (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(a)}{n!}(s-1)^n. \end{aligned}$$

The fractional Stieltjes constants generalize the Stieltjes constants. In [6, Corollary 3.2] we show

$$\gamma_\alpha(1) \to \gamma - 1 = -0.42278\,43350\ldots \text{ as } \alpha \to 0^+, \tag{2}$$

where $\gamma = \gamma_0 = \gamma_0(1) = 0.57721\,46649\ldots$ is Euler's constant. In [6] we also prove a conjecture of Kreminski [11, Conjecture IIIa]: Let $0 < \alpha \in \mathbb{R}$ and

$$C_\alpha(a) := \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$$

and $h_a(s) := \zeta(s,a) - 1/(s-1) - 1/a^s$ then

$$C_\alpha(a) = (-1)^{-\alpha} D_s^\alpha[h_a](1).$$

The goal of this paper is to compute $\gamma_\alpha(a)$ by evaluating $C_\alpha(a)$ and to find an upper bound for $|C_\alpha(a)|$. We start by recalling and proving some results about Stirling numbers (section

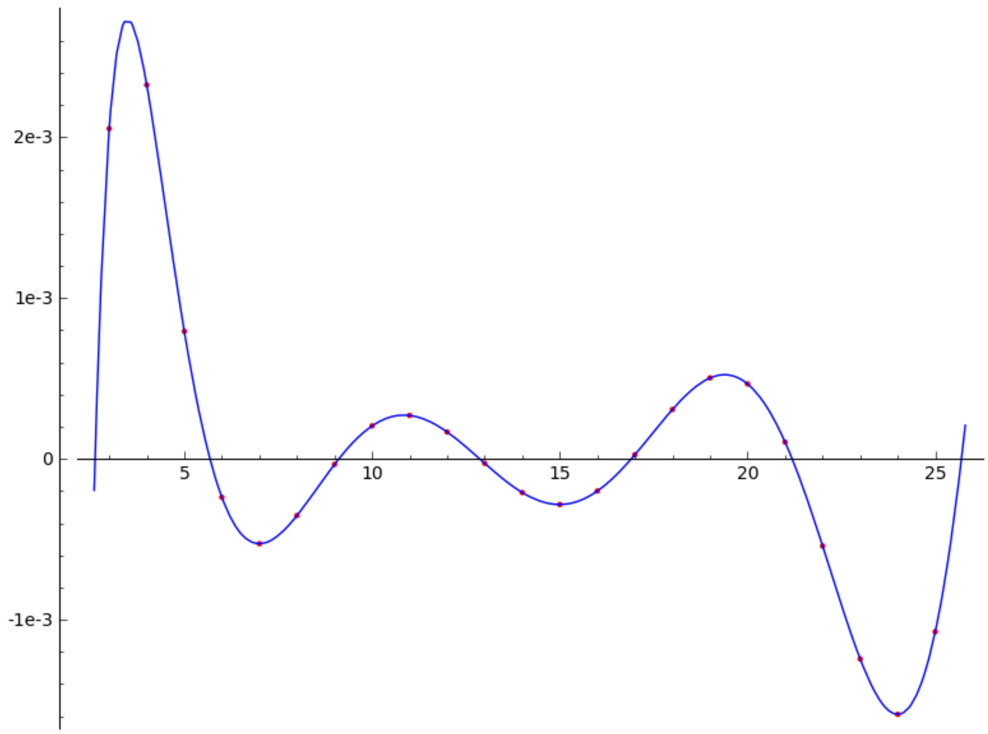

FIGURE 1. The fractional Stieltjes constants $\gamma_\alpha(1)$ plotted for $\alpha \in [2.6, 25.8]$ with (integral) Stieltjes constants (●).

2), that we employ in a method for evaluating $C_\alpha(a)$ (section 3). In section 4 we give an upper bound for $C_\alpha(a)$ for $\alpha > 1$ which is a generalization of [14, Theorem 3] to fractional Stieltjes constants and show how our bound can be minimized.

2. Stirling Numbers

For $\alpha \in \mathbb{R}$ let $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$ the falling factorial of α. We denote by $S(n,i,s)$ where $n \in \mathbb{N}_0$, $0 \le i \le n$ and $s \in \mathbb{C}$ the non-central Stirling numbers of the first kind, which satisfy the recurrence relations:

$$\begin{aligned} &S(0,0,s) = 1,\ S(1,0,s) = -s,\ S(1,1,s) = 1 \\ &S(n+1,0,s) = (-s-n)S(n,0,s),\ S(n+1,n+1,s) = S(n,n,s) \\ &S(n+1,i,s) = (-s-n)S(n,i,s) + S(n,i-1,s) \text{ for } 1 \le i \le n. \end{aligned} \tag{3}$$

The following is a generalization of results found in [5] and [9].

Lemma 2.1. *Let $\alpha \in \mathbb{R}$, $s \in \mathbb{C}$, and $g_\alpha(x) = \frac{\log^\alpha(x)}{x^s}$. For any $n \in \mathbb{N}_0$*

$$g_\alpha^{(n)}(x) = \sum_{i=0}^{n} S(n,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+n}}.$$

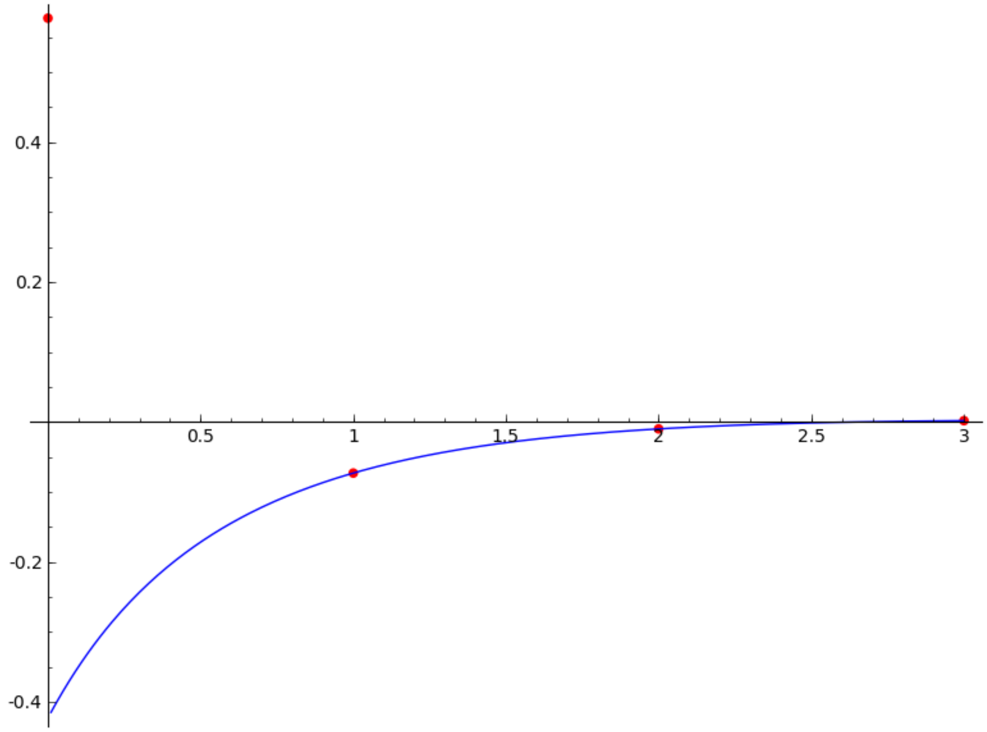

FIGURE 2. The fractional Stieltjes Constants $\gamma_\alpha(1)$ plotted for $\alpha \in [0,3]$ with (integral) Stieltjes constants (●). This plot illustrates the discontinuity of $\gamma_\alpha(1)$ at $\alpha = 0$. We have $\lim_{\alpha\to 0}\gamma_\alpha(1) = \gamma - 1$ (Equation (2)).

Proof. We proceed by way of induction on n. When $n = 0$, the result is trivially true. For $n = 1$ we have

$$g'_\alpha(x) = -s\frac{\log^\alpha(x)}{x^{s+1}} + \alpha\frac{\log^{\alpha-1}(x)}{x^{s+1}} = S(1,0,s)(\alpha)_0\frac{\log^\alpha(x)}{x^{s+1}} + S(1,1,s)(\alpha)_1\frac{\log^{\alpha-1}(x)}{x^{s+1}}.$$

Thus, the induction has been anchored. We now assume the result holds for some $k \in \mathbb{N}$. Differentiating $g_\alpha^{(k)}(x)$ we get

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=0}^{k}S(k,i,s)(\alpha)_i(\alpha-i)\frac{\log^{\alpha-i-1}(x)}{x^{s+k+1}} \\ &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=0}^{k}S(k,i,s)(\alpha)_{i+1}\frac{\log^{\alpha-i-1}(x)}{x^{s+k+1}}. \end{aligned}$$

Making a change of variables in the second sum yields

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= \sum_{i=0}^{k}(-s-k)S(k,i,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} + \sum_{i=1}^{k+1}S(k,i-1,s)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \\ &= (-s-k)S(k,0,s)(\alpha)_0\frac{\log^{\alpha}(x)}{x^{s+k+1}} + \sum_{i=1}^{k+1}\left((-s-k)S(k,i,s) + S(k,i-1,s)\right)(\alpha)_i\frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \end{aligned}$$

With the recurrence relation (3) we obtain $S(k+1,0,s) = (-s-k)S(k,0,s)$ and $S(k+1,i,s) = (-s-k)S(k,i,s) + S(k,i-1,s)$. Hence, we have

$$\begin{aligned} g_\alpha^{(k+1)}(x) &= S(k+1,0,s)(\alpha)_0 \frac{\log^\alpha(x)}{x^{s+k+1}} + \sum_{i=1}^{k+1} S(k+1,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k+1}} \\ &= \sum_{i=0}^{k+1} S(k+1,i,s)(\alpha)_i \frac{\log^{\alpha-i}(x)}{x^{s+k+1}}. \end{aligned}$$

Thus the relation holds for $g_\alpha^{(k+1)}(x)$. Hence, by induction, the lemma is proven. □

Recall that the (signed) Stirling numbers $s(i,j)$ of the first kind are generated by the recurrence:

$$\begin{aligned} &s(0,0) = 1,\ s(n,0) = s(0,n) = 0 \text{ for } n \in \mathbb{N} \\ &s(n+1,i) = -ns(n,i) + s(n,i-1) \text{ for } n \in \mathbb{N}_0 \text{ and } i \in \mathbb{N}_0 \end{aligned} \tag{4}$$

Proposition 2.2. *Let $\alpha \geq 0$, $0 < a \leq 1$, and $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$. Then for any $n \in \mathbb{N}_0$,*

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1,i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+a)}{(x+a)^{n+1}}. \tag{5}$$

Proof. In view of Lemma 2.1, the result is proven if we show that for all $n \in \mathbb{N}_0$ and all integers $0 \leq i \leq n$ we have $S(n,i,1) = s(n+1,i+1)$. We prove this equality by induction on n,

For $n = 0$ we get from the recurrence relation (3) that $S(0,0,1) = 1$. From (4) we get $s(1,1) = 1$. Thus, the induction is anchored. Now let $n \in \mathbb{N}$ and assume for all integers $0 \leq r \leq n$, $S(r,i,1) = s(r+1,i+1)$ for $i = 0,1,\ldots,r$.

Next we show $S(n+1,i,1) = s(n+2,i+1)$. With the recurrence relations (3) and (4), and the induction hypothesis we obtain

$$\begin{aligned} &S(n+1,0,1) = (-n-1)S(n,0,1) = -(n+1)s(n+1,1) = s(n+2,1) \\ &S(n+1,n+1,1) = S(n,n,1) = s(n+1,n+1). \end{aligned}$$

For $1 \leq i \leq n$ we have

$$\begin{aligned} S(n+1,i,1) &= (-n-1)S(n,i,1) + S(n,i-1,1) \\ &= -(n+1)s(n+1,i+1) + s(n+1,i) = s(n+2,i+1). \end{aligned}$$

Thus, by induction the result has been proven. □

3. Evaluation Of $\gamma_\alpha(a)$

To evaluate $\gamma_\alpha(a)$ we approximate $C_\alpha(a)$ and then use that $\gamma_\alpha(a) = C_\alpha(a) + \frac{\log^\alpha(a)}{a}$. Let $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$. By [6, Theorem 3.1] for $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$ and $m \in \mathbb{N}$ we have

$$\gamma_\alpha(a) = \sum_{r=0}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int_m^\infty P_1(x) f_\alpha'(x)dx, \tag{6}$$

where $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$. Thus

$$(7) \qquad C_\alpha(a) = \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} + \int\limits_m^\infty P_1(x) f'_\alpha(x) dx.$$

Integrating by parts $v \in \mathbb{N}$ times yields

$$(8) \qquad \int_m^\infty P_1(x) f'_\alpha(x) dx = \sum_{j=1}^{v} \left[P_j(x) f_\alpha^{(j-1)}(x) \right]_{x=m}^\infty + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx$$

where $P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}$ is the k^{th} periodic Bernoulli polynomial and B_j is the j^{th} Bernoulli number. For computational purposes, it is useful to recall that $B_j = 0$ for j odd.

As we will soon see, letting $m > 0$ forces the integral on the right hand side of (8) to converge for any $v \in \mathbb{N}$. With Proposition 2.2, we see that as $x \to \infty$, $f_\alpha^{(n)}(x) \to 0$ for any $n \in \mathbb{N}$. Thus, we can write (8) as

$$(9) \qquad \int_m^\infty P_1(x) f'_\alpha(x) dx = -\sum_{j=1}^{v} P_j(m) f_\alpha^{(j-1)}(m) + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx.$$

For any $j \in \mathbb{N}$ and $m \in \mathbb{N}$ we have $P_j(m) = \frac{B_j}{j!}$. We now approximate $C_\alpha(a)$ by

$$(10) \qquad C_\alpha(a) \approx \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} - \sum_{j=1}^{v} \frac{B_j}{j!} f_\alpha^{(j-1)}(m).$$

The error in approximating $C_\alpha(a)$ by (10) is given by $R_v = (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx$. We now show that we can choose m and v so that the error is arbitrarily small. We choose $v > 1$. As $|P_n(x)| \leq \frac{3+(-1)^n}{(2\pi)^n}$ for any $n > 1$ (see [14], [3], or [13]) we have

$$(11) \qquad |R_v| = \left| (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) \right| \leq \frac{3+(-1)^n}{(2\pi)^n} \int\limits_m^\infty \left| f_\alpha^{(v)}(x) \right| dx.$$

With Corollary 2.2 and the triangle inequality in (11) we get

$$(12) \qquad |R_v| \leq \frac{3+(-1)^v}{(2\pi)^v} \sum_{i=0}^{v} |s(v+1, i+1)| \frac{\Gamma(\alpha+1)}{|\Gamma(\alpha - i + 1)|} \int\limits_m^\infty \frac{\log^{\alpha-i}(x+a)}{(x+a)^{v+1}} dx.$$

We write the integral in terms of the Upper Incomplete Gamma function (see [7, p. 346] and [1, 6.5.3])

$$(13) \qquad \int\limits_m^\infty \frac{\log^{\alpha-i}(x+a)}{(x+a)^{v+1}} dx = \frac{\Gamma(\alpha - i + 1, v \log(m+a))}{v^{\alpha-i+1}}.$$

Applying (13) in (12) we find an upper bound for the error:

$$(14) \qquad |R_v| \leq \frac{(3+(-1)^v)\Gamma(\alpha+1)}{(2\pi)^v v^{\alpha+1}} \sum_{i=0}^{v} |s(v+1, i+1)| \frac{\Gamma(\alpha - i + 1, v \log(m+a)) v^i}{|\Gamma(\alpha - i + 1)|}.$$

The error term, R_{2v}, in (12) converges for all v. To find suitable parameters v and m so that R_{2v} we follow a similar method to that used to evaluate $\zeta^{(k)}$ discussed in [5]. We first let v be large and then iteratively increase the value of m until the error is small as desired. To illustrate the method, letting $v = 101$ (this value was also used in [5]), we evaluate the bound (12) for $N = 200, 300, \ldots$ until the error is as small as desired. For example, if $\alpha = 100$, $v = 101$, and $N = 200$, then $|R_{2v}| < 1.769892 \cdot 10^{-100}$. If $N = 1500$ then $|R_{2v}| < 1.245704 \cdot 10^{-253}$.

We have shown:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$ with $\alpha > 0$, $0 < a \leq 1$, $m \in \mathbb{N}$, and $v > 1$. Let*

$$C'_\alpha(a) := \sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} - \sum_{j=1}^{v} \frac{B_j}{j!} f_\alpha^{(j-1)}(m).$$

Then

$$|C'_\alpha(a) - C_\alpha(a)| \leq \frac{(3+(-1)^v)\Gamma(\alpha+1)}{(2\pi)^v v^{\alpha+1}} \sum_{i=0}^{v} |s(v+1, i+1)| \frac{\Gamma(\alpha - i + 1, v\log(m+a))v^i}{|\Gamma(\alpha - i + 1)|}.$$

The method described was implemented in the C library, Arb. At a later date, this method will be included in the Arb library. The values for $\gamma_\alpha(a)$ in Figures 1 and 2 were computed using the method described.

4. An Upper Bound For $C_\alpha(a)$

Using $m = 1$ in (6), we have after making some minor simplifications

$$\gamma_\alpha(a) = \frac{\log^\alpha(a)}{a} + \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1} + \int_1^\infty P_1(x) f'_\alpha(x)dx \tag{15}$$

Since $0 < a \leq 1$ and $P_1(x) = x - \frac{1}{2}$ on $(0,1)$ integration by parts yields

$$\int_{1-a}^{1} P_1(x) f'_\alpha(x)dx = \int_{1-a}^{1} \left(x - \frac{1}{2}\right) f'_\alpha(x)dx = \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1}$$

Using this in (15), allows us to see that

$$\gamma_\alpha(a) = \frac{\log^\alpha(a)}{a} + \int_{1-a}^{\infty} P_1(x) f'_\alpha(x)dx = \frac{\log^\alpha(a)}{a} + C_\alpha(a).$$

Using corollary 2.2 we have for any positive integer n,

$$f_\alpha^{(n)}(x) = \sum_{i=0}^{n} s(n+1, i+1)(\alpha)_i \frac{\log^{\alpha-i}(x+a)}{(x+a)^{n+1}}. \tag{16}$$

Assume $\alpha > 1$, let n be any arbitrary integer satisfying $1 \leq n < \alpha$, and let k be any positive integer so that $1 \leq k \leq n$. From these assumptions, we see that $f_\alpha^{(k)}(x-a)$ is a combination of positive powers of $\log(x)$ hence, $f_\alpha^{(k)}(1-a) = 0$. Also, $f_\alpha^{(k)}(x-a) \to 0$ as $x \to \infty$. These observations and integrating by parts n times yield

$$\begin{aligned} C_\alpha(a) &= P_2(x) f'_\alpha(x)\big|_{x=1-a}^{\infty} - P_3(x) f''_\alpha(x)\big|_{x=1-a}^{\infty} + \ldots + (-1)^{n+1} P_{n+1}(x) f_\alpha^{(n)}(x)\big|_{x=1-a}^{\infty} \\ &\quad + (-1)^n \int_{1-a}^{\infty} P_{n+1}(x) f_\alpha^{(n+1)}(x)\,dx \\ &= (-1)^n \int_{1-a}^{\infty} P_{n+1}(x) f_\alpha^{(n+1)}(x)\,dx. \end{aligned}$$

Making a change of variable we get

$$C_\alpha(a) = (-1)^n \int_1^{\infty} P_{n+1}(x-a) f_\alpha^{(n+1)}(x-a)\,dx.$$

Knopp showed in [10] that $|P_n(x)| \le \frac{4}{(2\pi)^n}$ for all integers $n > 1$. Ostrowski observed in [13] that for odd $n > 1$, $|P_n(x)| < \frac{2}{(2\pi)^n}$. Thus we can write $|P_n(x)| \le \frac{3+(-1)^n}{(2\pi)^n}$ for all $n > 1$. Making use of this inequality, we now derive an upper bound for $C_\alpha(a)$. We have

$$\begin{aligned} |C_\alpha(a)| &= \left| (-1)^n \int_1^{\infty} P_{n+1}(x-a) f_\alpha^{(n+1)}(x-a)\,dx \right| \\ &\le \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \int_1^{\infty} \left| f_\alpha^{(n+1)}(x-a) \right| dx \\ &\le \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2, i+1)| (\alpha)_i \int_1^{\infty} \frac{\log^{\alpha-i}(x)}{x^{n+2}}\,dx. \end{aligned} \tag{17}$$

We now evaluate the integral in (17). After a change of variables we have

$$\int_1^{\infty} \frac{\log^{\alpha-i}(x)}{x^{n+2}}\,dx = \frac{1}{(n+1)^{\alpha-i+1}} \int_0^{\infty} x^{\alpha-i} e^{-x}\,dx = \frac{\Gamma(\alpha-i+1)}{(n+1)^{\alpha-i+1}}, \tag{18}$$

since $\alpha - i \ge \alpha - n > 0$, and the integral converges for all $0 \le i \le n+1$. Using this in (17),

$$|C_\alpha(a)| \le \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2, i+1)| (\alpha)_i \frac{\Gamma(\alpha-i+1)}{(n+1)^{\alpha-i+1}}. \tag{19}$$

Since $1 \le n < \alpha$, we can write $(\alpha)_i = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)}$ for each $0 \le i \le n+1$. From (19) we get

$$\begin{aligned}
|C_\alpha(a)| &\le \frac{3+(-1)^{n+1}}{(2\pi)^{n+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)| \frac{\Gamma(\alpha+1)}{(n+1)^{\alpha-i+1}} \\
&= \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \sum_{i=0}^{n+1} |s(n+2,i+1)|(n+1)^i \\
&= \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+2}} \sum_{j=1}^{n+2} |s(n+2,j)|(n+1)^j.
\end{aligned}$$

By [14, 6.14] we have $\sum_{i=1}^{n+2} |s(n+2,j)|(n+1)^j = \frac{(2n+2)!}{n!}$. Using this identity, we arrive at

$$|C_\alpha(a)| \le \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+2}} \frac{(2n+2)!}{n!} = \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!}.$$

We have proven:

Theorem 4.1. *Let $0 < a \le 1$, $\alpha > 1$ and $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$. Then,*

$$|C_\alpha(a)| \le \frac{(3+(-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!} \tag{20}$$

where n is any positive integer satisfying $1 \le n < \alpha$.

We now improve Theorem 4.1. The first step is to notice that the inequality holds for any positive integer n with $1 \le n < \alpha$. It is natural to wonder what value of n minimizes the upper bound. The Lambert-W function – the complex values $W(z)$ for which $W(z)e^{W(z)} = z$ – will help us establish this, together with the following bound: For all $n \ge 1$,

$$\frac{(2n)!}{n!} \le \sqrt{2}\left(\frac{4n}{e}\right)^n e^{\frac{1}{24n} - \frac{1}{12n+1}} < \sqrt{2}\left(\frac{4n}{e}\right)^n, \tag{21}$$

which follows directly from the following sharp version of Stirling's formula:

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{1}{12n+1}} \le n! \le \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{1}{12n}}.$$

In particular, we get:

Theorem 4.2. *Let $0 < a \le 1$ and $\alpha > 0$. Let n be chosen in the following manner: if $\frac{\pi}{2}e^{W(\frac{2(\alpha+1)}{\pi})} < \alpha$, then let n be the nearest integer to $\frac{\pi}{2}e^{W(\frac{2(\alpha+1)}{\pi})}$. Otherwise, let n be the greatest integer not exceeding α. Choosing n in this way makes the right hand side of the inequality in theorem 4.1 smallest of all the possible choices.*

Proof. We apply (21) to the right hand side of the inequality in theorem 4.1, and take $g(x) = \frac{4\sqrt{2}\Gamma(\alpha+1)}{x^{\alpha+1}}\left(\frac{2n}{e\pi}\right)^x$. It is our goal to find x on the closed interval $[1,\alpha]$ that minimizes $g(x)$. Once x is found, we let n be the nearest integer to x so that $g(n)$ is smallest. Let $\tilde{C}_\alpha = 4\sqrt{2}\Gamma(\alpha+1)$. Since we are working on a closed interval and g is continuous on $[1,\alpha]$, g must attain a minimum on $[1,\alpha]$. We first find the derivative of $g(x)$ by observing

$$g(x) = \frac{\tilde{C}_\alpha}{x^{\alpha+1}}\left[\frac{2x}{\pi e}\right]^x = \tilde{C}_\alpha \exp\left[-(\alpha+1)\log(x) + x\log\left(\frac{2x}{\pi e}\right)\right].$$

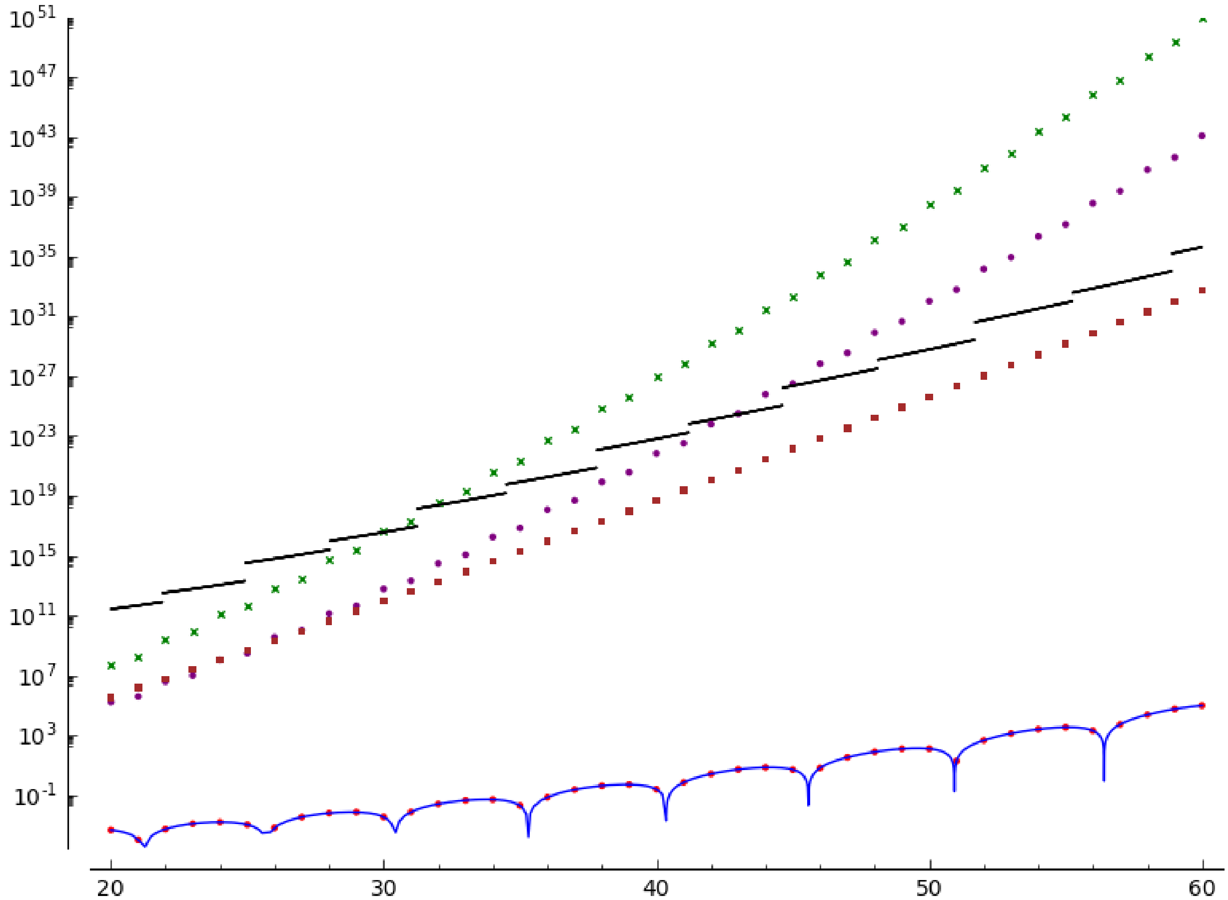

FIGURE 3. The absolute value of the fractional Stieltjes constants (–) $\gamma_\alpha(a)$ for $20 \leq \alpha \leq 60$; with integral Stieltjes constants (•); the bound (–) for the fractional Stieltjes constants from Theorem 4.2; the bound (x) by Berndt [3]; the bound (•) by William and Zhang [14]; the bound (■) by Matsuoka [12].

Differentiating, we find

$$g'(x) = \tilde{C}_\alpha \left[\frac{-(\alpha+1)}{x} + 1 + \log\left(\frac{2x}{\pi e}\right)\right] \exp\left[-(\alpha+1)\log(x) + x\log\left(\frac{2x}{\pi e}\right)\right].$$

Setting $g'(x) = 0$ dividing both sides by the constant term and the exponential term, we get

$$\frac{-(\alpha+1)}{x} + 1 + \log\left(\frac{2x}{\pi e}\right) = \frac{-(\alpha+1)}{x} + \log\left(\frac{2x}{\pi}\right) = 0.$$

This implies that $\frac{2x}{\pi}\log\left(\frac{2x}{\pi}\right) = \frac{2(\alpha+1)}{\pi}$, and if we let $y = \log\left(\frac{2x}{\pi}\right)$, then the previous equation becomes $ye^y = \frac{2(\alpha+1)}{\pi}$. Applying the Lambert-W function, we see that we must have $y = e^{W\left(\frac{2(\alpha+1)}{\pi}\right)}$. Solving for x, using this relation we then have $x = \frac{\pi}{2}e^{W\left(\frac{2(\alpha+1)}{\pi}\right)}$. If $x \leq \alpha$, then naturally we should pick n to be the greatest integer not exceeding α. This is because this would imply that $g(x)$ is monotonically decreasing on the interval $[1, \alpha]$. If x falls within the closed interval $[1, \alpha]$, then we pick the closest integer to x. This proves the result. $\square$

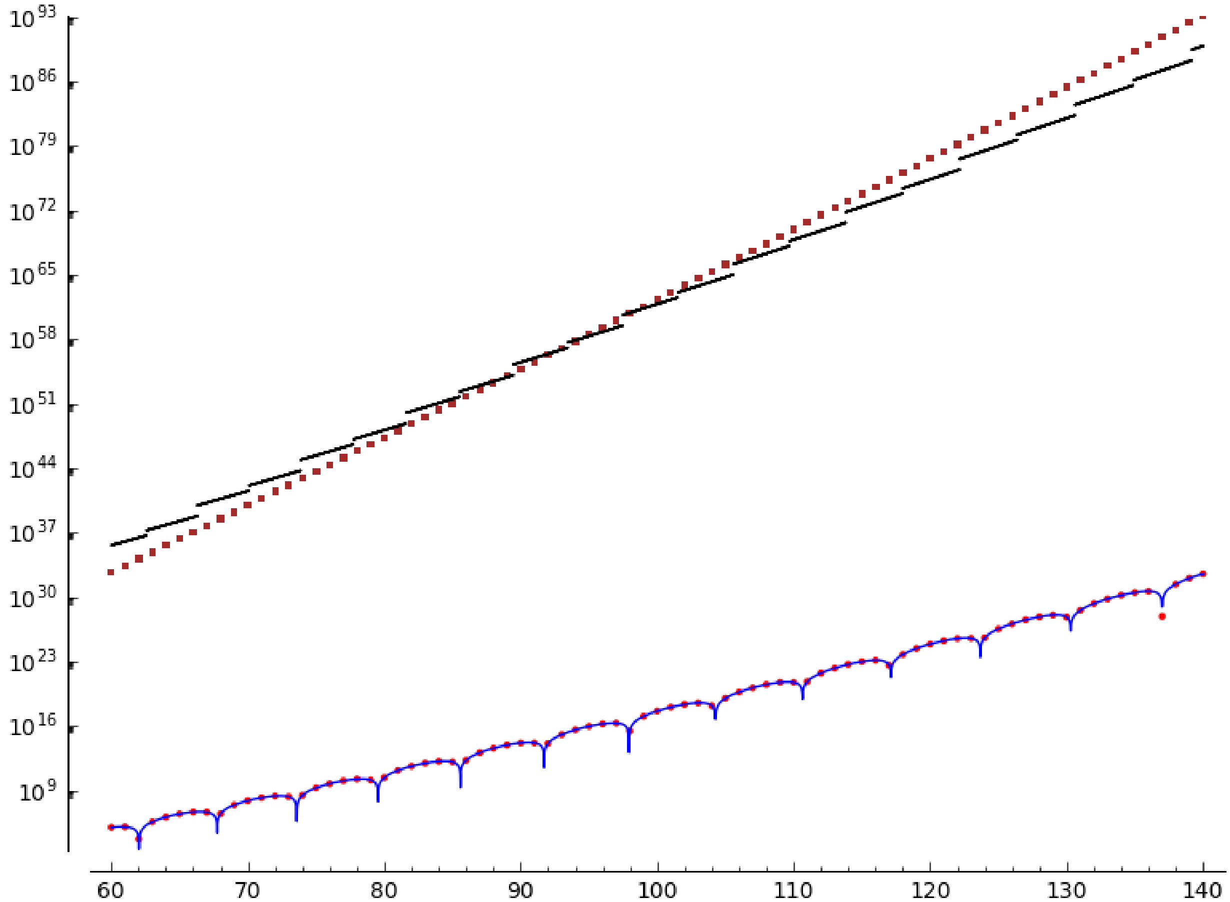

FIGURE 4. The absolute value of the fractional Stieltjes constants (–) $\gamma_\alpha(a)$ for $60 \leq \alpha \leq 140$; with integral Stieltjes constants (●); the bound (–) for the fractional Stieltjes constants from Theorem 4.2; the bound (■) by Matsuoka [12].

The upper bound for the fractional Stieltjes constants yields a bound for the integral Stieltjes constants. In Figures 3 and 4 we compare our bound to previously known bounds for integral Stieltjes Constants. Namely the bound by Berndt [3]

$$\gamma_m = \gamma_m(1) \leq \frac{(3+(-1)^m)(m-1)!}{\pi^m}$$

and the bound by Williams and Zhang [14]

$$\gamma_m = \gamma_m(1) \leq \frac{(3+(-1)^m)(2m)!}{m^{m+1}(2\pi)^m}$$

and the bound by Matsuoka [12]

$$\gamma_m = \gamma_m(1) < 10^{-4}(\log m)^m \text{ for } n > 4.$$

Remark 4.3. Theorem 4.1 with $n+1 = m$ and $\alpha = n$ yields the bound by William and Zhang.

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