THE FIRST DIGIT OF THE DISCRIMINANT OF EISENSTEIN POLYNOMIALS AS AN INVARIANT OF TOTALLY RAMIFIED EXTENSIONS OF P-ADIC FIELDS

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Abstract. Let K be an extension of the p-adic numbers with uniformizer π. Let φ and ψ be Eisenstein polynomials over K of degree n that generate isomorphic extensions. We show that, if the cardinality of the residue class field of K divides $n(n-1)$ then $v(\mathrm{disc}(f) - \mathrm{disc}(g)) > v(discf)$. This makes the first (non-zero) digit of the π-adic expansion of $\mathrm{disc}(f)$ an invariant of the extension generated by φ. Furthermore we find that non-cyclic extensions of degree p of the field of p-adic numbers are uniquely determined by this invariant.

1. Introduction

For a field extension of finite degree the discriminant of an integral basis yields invariants of the extension. A change of integral basis results in the multiplication of the discriminant by the square of a unit in the base ring, namely by the determinant of the transformation matrix. So, in the case of extensions of $\mathbb{Q}$, as the only units in $\mathbb{Z}$ are -1 and 1, the discriminant is an invariant of the extension.

In the case of local fields the valuation of the discriminant is used as an invariant. For a formulation of the results described above for local fields see [Cas86, chapter 7, section 6].

Let L be a totally ramified extension of a finite extension K of $\mathbb{Q}_p$. A power integral basis of the valuation ring $\mathcal{O}_L$ over the valuation ring $\mathcal{O}_K$ of K consists of the powers $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ of a uniformizer α where n is the degree of the extension L/K. The discriminant of such a basis is the discriminant of the Eisenstein polynomial φ with root α. Denote by π a uniformizer of the base ring $\mathcal{O}_K$ of $\mathcal{O}_L$. We investigate the use of the first digits of the π-adic expansion of the discriminant of the generating Eisenstein polynomials as invariants of totally ramified extensions.

Notation. For a prime number p we denote by $\mathbb{Q}_p$ the completion of the rational numbers $\mathbb{Q}$ with respect to the p-adic exponential valuation v_p, by $\mathbb{Z}_p$ its valuation ring, and by $\overline{\mathbb{Q}_p}$ a fixed algebraic closure of $\mathbb{Q}_p$. By K we denote a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$, uniformizing element π, and maximal ideal (π). The exponential valuation v_π on K is normalized such that $v_\pi(\pi) = 1$. The extensions of v_p and v_π to $\overline{\mathbb{Q}_p}$ are also denoted by v_p and v_π. We write $\underline{K}$ for the residue class field $\mathcal{O}_K/(\pi)$ of K and for $\beta \in \mathcal{O}_K$ we set $\underline{\beta} = \beta + (\pi) \in \underline{K}$.

2. Main Results

In this section we present our main results. The proofs can be found in the following sections.

The extensions that we consider are either totally ramified extensions of $\mathbb{Q}_p$ or of a finite extension K of $\mathbb{Q}_p$. Such an extension L can be generated by an Eisenstein polynomial φ of degree n. For a root α of φ we have $\mathcal{O}_L = \mathcal{O}_K(\alpha)$. The discriminant $\mathrm{disc}(\varphi)$ of the generating polynomial φ is equal to the discriminant of the integral basis $1, \alpha, \ldots, \alpha^{n-1}$ of $\mathcal{O}_L/\mathcal{O}_K$. The root α is a uniformizing element of $\mathcal{O}_L$. We write v_α for the valuation on $\mathcal{O}_L$ that is normalized such that $v_\alpha(\alpha) = 1$. We investigate the use of the digits of the π-adic expansion (or the p-adic expansion when considering extensions of $\mathbb{Q}_p$) of the discriminant of generating polynomials of totally ramified extensions as invariants of the extension.

When we choose two polynomials φ and ψ that generate isomorphic extensions such that they are close enough, that is $d(\varphi, \psi)$ in the following theorem is large enough, we can assure that arbitrarily many digits of the π-adic expansion of their discriminants coincide.

Theorem 2.1. Let $\varphi(x) = \sum_{i=0}^{n} \varphi_i x^i \in \mathcal{O}_K[x]$ and $\psi(x) = \sum_{i=0}^{n} \psi_i x^i \in \mathcal{O}_K[x]$ be Eisenstein of degree n with $v_\pi(\mathrm{disc}(\varphi)) = v_\pi(\mathrm{disc}(\psi)) = n + j - 1$. Let the distance between φ and ψ be given by $d(\varphi, \psi) = \min_{0 \le i \le n-1}\{v_\pi(\psi_i - \varphi_i) + \frac{i}{n}\}$. If $d(\varphi, \psi) > \frac{n+2j}{n}$ then

$$v_\pi(\mathrm{disc}(\varphi) - \mathrm{disc}(\psi)) \geq \left(1 - \frac{1}{n}\right) v_\pi(\mathrm{disc}(\varphi)) + d(\varphi, \psi).$$

From now on we concentrate on the first digit of the π-adic expansion of the discriminants of polynomials. We use this notation:

Definition 2.2. For $\beta \in \mathcal{O}_K$ we set $\mathrm{tc}\,(\beta) = \underline{(\beta/\pi^{v_\pi(\beta)})} \in \underline{K}$. This is the first nonzero digit (or the trailing coefficient) of the π-adic expansion of β as an element of $\underline{K}$.

Clearly we have:

Lemma 2.3. For $\alpha, \beta \in K$ we have $\mathrm{tc}\,(\alpha \cdot \beta) = \mathrm{tc}\,(\alpha) \cdot \mathrm{tc}\,(\beta)$.

We find that this first π-adic digit of the valuation of the generating Eisenstein polynomial is an invariant in many cases.

Theorem 2.4. Let $\varphi, \psi \in \mathcal{O}_K[x]$ be Eisenstein and of degree n such that $K[x]/(\varphi) \cong K[x]/(\psi)$. If $(\#\underline{K} - 1) | (n(n-1))$ then $\mathrm{tc}\,(\mathrm{disc}(\varphi)) = \mathrm{tc}\,(\mathrm{disc}(\psi))$.

In these cases, the trailing coefficient of the discriminant is independent of the generating Eisenstein polynomial and thus is an invariant of power integral bases of a totally ramified extension. Thus, if $(\#\underline{K} - 1) | (n(n-1))$, then the trailing coefficient of the discriminant is an invariant of the extension.

Some classes of extensions always have the same invariant.

Proposition 2.5. Let p be odd and $\varphi \in \mathbb{Q}_p[x]$ be Eisenstein of degree p^m such that $\mathbb{Q}_p[x]/(\varphi)$ is cyclic. Then

$$\mathrm{tc}\,(\mathrm{disc}(\varphi)) = \begin{cases} 1 & \text{if } m \text{ is even} \\ -1 & \text{if } m \text{ is odd and } p \equiv 1 \bmod 4 \\ 1 & \text{if } m \text{ is odd and } p \equiv 3 \bmod 4 \end{cases}$$

We end with examples of information that can be obtained from this invariant. Extensions of $\mathbb{Q}_p$ of degree p have been described in detail, see [Ama71] or [JR06]. We show that they can also be classified using the discriminant and its trailing coefficient, that is, the first non-zero coefficient of its p-adic expansion.

Theorem 2.6. Let φ be an Eisenstein polynomial of degree p in $\mathbb{Q}_p$ such that $\mathrm{Gal}(\varphi) \ncong C_p$ and $v_p(\mathrm{disc}(\varphi)) \neq 2p-1$. Then the isomorphism class of the extension generated by φ is uniquely determined by $v_p(\mathrm{disc}(\varphi))$ and $\mathrm{tc}\,(\mathrm{disc}(\varphi))$.

The Galois group of an Eisenstein polynomial φ of degree p over $\mathbb{Q}_p$ can be determined from the valuation of its discriminant $\mathrm{disc}(\varphi)$ and its trailing coefficient $\mathrm{tc}\,(\mathrm{disc}(\varphi))$.

Corollary 2.7. Let p be an odd prime, and let φ be an Eisenstein polynomial of degree p over $\mathbb{Q}_p$. Let $v = v_p(\mathrm{disc}(\varphi))$, $j = v-p+1$, and $\underline{\gamma} = (\underline{-1})^{(p-1)/2}\,\mathrm{tc}\,(\mathrm{disc}(\varphi))$. Then

$$\mathrm{Gal}(\varphi) \cong \begin{cases} C_p \rtimes C_{p-1} & \text{if } v = 2p-1 \\ C_p & \text{if } v = 2p-2 \text{ and } \underline{\gamma} = \underline{p-1} \\ C_p \rtimes C_d & \text{otherwise} \end{cases}$$

where $d = \frac{p-1}{\gcd(\frac{p-1}{m}, j)}$ with m being the order of $\underline{aj}$ in $\mathbb{F}_p^\times$ for $\underline{a} = \underline{\gamma} \cdot (\underline{-1})^{j+1}\underline{j}^{-1}$.

3. Proof of Theorem 2.1

The distance function d for Eisenstein polynomials from Theorem 2.1 was introduced by Marc Krasner for the proof for his mass formula for extensions of local fields, see [Kra66] or [PR01]. We recall some of the results from this work.

Assume that $\varphi(x) = \sum_{i=0}^n \varphi_i x^i \in \mathcal{O}_K[x]$ and $\psi(x) = \sum_{i=0}^n \psi_i x^i \in \mathcal{O}_K[x]$ are Eisenstein polynomials whose discriminants have the same valuation, say $v_\pi(\mathrm{disc}(\varphi)) = v_\pi(\mathrm{disc}(\psi)) = n+j-1$. If $\alpha = \alpha_{(1)}, \ldots, \alpha_{(n)}$ are the roots of φ in $\overline{\mathbb{Q}_p}$ and β is a root of ψ then

$$d(\varphi,\psi) = \min_{0\le i\le n-1}\left\{v_\pi(\psi_i - \varphi_i) + \frac{i}{n}\right\} = \sum_{i=1}^n \min\{v_\pi(\alpha-\beta), v_\pi(\alpha-\alpha_{(i)})\}.$$

If $d(\varphi,\psi) > \frac{n+2j}{n}$ then there is a root $\beta \in \overline{\mathbb{Q}_p}$ of ψ such that $v_\pi(\alpha-\beta) > v_\pi(\alpha-\alpha_{(i)})$ for $2 \le i \le n$. In this case Krasner's lemma implies that $K(\alpha) = K(\beta)$. So the assumption of Theorem 2.1 implies that the extensions generated by φ and ψ are isomorphic.

Furthermore, when $v_\pi(\alpha-\beta) > v_\pi(\alpha-\alpha_{(i)})$ for $2 \le i \le n$, we obtain another expression for the distance of two polynomials:

$$d(\varphi,\psi) = \sum_{i=1}^n \min\{v_\pi(\alpha-\beta), v_\pi(\alpha-\alpha_{(i)})\} = v_\pi(\alpha-\beta) + \sum_{i=2}^n v_\pi(\alpha-\alpha_{(i)})$$

So we can write the valuation of $\alpha-\beta$ in terms of $d(\varphi,\psi)$ and $\mathrm{disc}(\varphi)$:

$$v_\pi(\alpha-\beta) = d(\varphi,\psi) - \sum_{i=2}^n v_\pi(\alpha-\alpha_{(i)}) \tag{1}$$

$$= d(\varphi,\psi) - v_\pi\left(\prod_{i=2}^n (\alpha-\alpha_{(i)})\right) \tag{2}$$

$$= d(\varphi,\psi) - \frac{1}{n} v_\pi(\mathrm{disc}(\varphi))$$

We now are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\alpha = \alpha_{(1)}, \ldots, \alpha_{(n)}$ as above. As $K(\alpha) = K(\beta)$ there are $m = v_\alpha(\alpha - \beta)$ and $\gamma \in K(\alpha)$ with $v_\alpha(\gamma) = 0$ such that $\beta = \alpha + \gamma\alpha^m$. We order the roots $\beta_{(1)}, \ldots, \beta_{(n)}$ of ψ such that $\beta_{(1)} = \beta = \alpha + \gamma\alpha^m$, $\beta_{(2)} = \alpha_{(2)} + \gamma_{(2)}\alpha_{(2)}^m$ and so on. For the discriminant of ψ we get:

$$\begin{aligned}
\operatorname{disc}(\psi) &= \prod_{i<j} \left(\beta_{(i)} - \beta_{(j)}\right)^2 = \prod_{i<j} \left(\left(\alpha_{(i)} + \gamma_{(i)}\alpha_{(i)}^m\right) - \left(\alpha_{(j)} + \gamma_{(j)}\alpha_{(j)}^m\right)\right)^2 \\
&= \prod_{i<j} \left(\left(\alpha_{(i)} - \alpha_{(j)}\right) + \left(\gamma_{(i)}\alpha_{(i)}^m - \gamma_{(j)}\alpha_{(j)}^m\right)\right)^2 \\
&= \prod_{i<j} \left(\alpha_{(i)} - \alpha_{(j)}\right)^2 \prod_{i<j} \left(1 - \sum_{k=1}^{m} \left(\gamma_{(j)}\alpha_{(i)}^{m-k}\alpha_{(j)}^{k-1}\right) - \alpha_{(i)}^m \frac{\gamma_{(j)} - \gamma_{(i)}}{\alpha_{(j)} - \alpha_{(i)}}\right)^2 \\
&= \operatorname{disc}(\varphi) \prod_{i<j} \left(1 - \gamma_{(j)} \sum_{k=1}^{m} \left(\alpha_{(i)}^{m-k}\alpha_{(j)}^{k-1}\right) - \alpha_{(i)}^m \frac{\gamma_{(j)} - \gamma_{(i)}}{\alpha_{(j)} - \alpha_{(i)}}\right)^2
\end{aligned}$$

Let $C_{ij} = \left(1 - \gamma_{(j)} \sum_{k=1}^{m} \left(\alpha_{(i)}^{m-k}\alpha_{(j)}^{k-1}\right) - \alpha_{(i)}^m \frac{\gamma_{(j)} - \gamma_{(i)}}{\alpha_{(j)} - \alpha_{(i)}}\right)^2$. We have $v_\alpha(C_{ij} - 1) \geq m - 1$. With $\operatorname{disc}(\psi) = \operatorname{disc}(\varphi)(\prod_{i<j} C_{ij})$ we get:

$$\begin{aligned}
v_\pi(\operatorname{disc}(\varphi) - \operatorname{disc}(\psi)) &= v_\pi\left(\operatorname{disc}(\varphi) - \operatorname{disc}(\varphi)\left(\prod_{i<j} C_{ij}\right)\right) \\
&= v_\pi(\operatorname{disc}(\varphi)) + v_\pi\left(1 - \left(\prod_{i<j} C_{ij}\right)\right) \\
&\geq v_\pi(\operatorname{disc}(\varphi)) + \frac{m-1}{n}.
\end{aligned}$$

With $m = v_\alpha(\alpha - \beta) = n \cdot v_\pi(\alpha - \beta)$ and (2) we obtain

$$\begin{aligned}
v_\pi(\operatorname{disc}(\varphi) - \operatorname{disc}(\psi)) &\geq v_\pi(\operatorname{disc}(\varphi)) + \frac{n \cdot v_\pi(\alpha - \beta) - 1}{n} \\
&= v_\pi(\operatorname{disc}(\varphi)) + d(\varphi, \psi) - \frac{1}{n}(v_\pi(\operatorname{disc}(\varphi)) + 1).
\end{aligned}$$

□

4. Proof of Theorem 2.4

Proof of Theorem 2.4. Let α be a root of φ. Since φ and ψ generate isomorphic extensions, there exists $\beta \in K(\alpha)$ such that $\psi(\beta) = 0$. So $\beta = \sum_{k=0}^{n-1} b_k\alpha^k$ for some $b_k \in \mathcal{O}_K$. As $v_\alpha(\beta) = v_\alpha(\alpha = 1)$ we have $v(b_1) = 0$. Let $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(n)}$ be the conjugates of α and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the isomorphisms such that $\sigma_i(\alpha) = \alpha_{(i)}$. Let $\beta_{(1)}, \beta_{(2)}, \ldots, \beta_{(n)}$ be the roots of ψ, defined by

$$\beta_{(i)} = \sigma_i(\beta) = \sigma_i\left(\sum_{k=0}^{n-1} b_k\alpha^k\right) = \sum_{k=0}^{n-1} \sigma_i(b_k)\sigma_i(\alpha)^k = \sum_{k=0}^{n-1} b_k\alpha_{(i)}^k.$$

We will now compute the discriminant of ψ, with the goal of writing it in terms of the discriminant of φ.

$$\begin{aligned}\operatorname{disc}(\psi) &= \prod_{i<j}\left(\beta_{(i)}-\beta_{(j)}\right)^2 = \prod_{i<j}\left(\sum_{k=1}^{n-1} b_k\left(\alpha_{(i)}^k-\alpha_{(j)}^k\right)\right)^2 \\ &= \prod_{i<j}\left(\alpha_{(i)}-\alpha_{(j)}\right)^2 \prod_{i<j}\left[\sum_{k=1}^{n-1}\left(b_k\sum_{\ell=0}^{k-1}\alpha_{(i)}^{(k-1-\ell)}\alpha_{(j)}^{\ell}\right)\right]^2 \\ &= \operatorname{disc}(\varphi)\cdot\prod_{i<j}\left[\sum_{k=1}^{n-1}\left(b_k\sum_{\ell=0}^{k-1}\alpha_{(i)}^{(k-1-\ell)}\alpha_{(j)}^{\ell}\right)\right]^2\end{aligned}$$

We can now write $\operatorname{disc}(\varphi)-\operatorname{disc}(\psi)=\operatorname{disc}(\varphi)\cdot(1-\gamma)$ where

$$\gamma = \prod_{i<j}\left[\sum_{k=1}^{n-1}\left(b_k\sum_{\ell=0}^{k-1}\alpha_{(i)}^{(k-1-\ell)}\alpha_{(j)}^{\ell}\right)\right]^2.$$

Note that γ is a symmetric polynomial in $\alpha_{(1)},\ldots,\alpha_{(n)}$. Let $e_1,\ldots,e_n$ refer to the elementary symmetric polynomials in $\alpha_{(1)},\ldots,\alpha_{(n)}$. By the fundamental theorem of symmetric polynomials, we can write γ as a polynomial γ^* in $(e_1,\ldots,e_n)$. If we expand γ^*, all terms have sums of products of $\alpha_{(1)},\ldots,\alpha_{(n)}$ except for the constant term, $b_1^{n(n-1)}$. So

$$\gamma^* = m_{(1)}e_1+m_{(2)}e_2+\cdots+m_{(n)}e_n+m_{(n+1)}e_1^2+m_{(n+2)}e_1e_2+\cdots+b_1^{n(n-1)}.$$

where $e_1,\ldots,e_n$ are the coefficients of φ. Since the coefficients of φ have π-adic valuation greater than or equal to 1, π divides all of its coefficients. This implies $\underline{\gamma}=\underline{b_1^{n(n-1)}}$.

The next step is to show $\underline{b_1^{n(n-1)}}=\underline{1}$. Since both φ and ψ are Eisenstein and generate the same extension, $v_\alpha(\beta)=1$. So

$$1=v_\alpha(\beta)=v_\alpha\left(\sum_{k=0}^{n-1}b_k\alpha^k\right)=\min_{0\le k\le n-1}\{v_\alpha(b_k\alpha^k)\}.$$

Equality holds because each b_k has α-adic valuation 0 or a positive multiple of n, so each term has a different valuation. For all k, $v_\alpha(b_k\alpha^k)\ge 1$. For $k\ge 1$, this is obviously true. For $k=0$, $v_\alpha(b_0)=0$ or a positive multiple of n. As $v_\alpha(b_0)$ must be at least 1, the lowest multiple of n it can be is n. We now have that for $k\neq 1$, $v_\alpha(b_k\alpha^k)\ge 2$. Thus $v_\alpha(b_1\alpha)=1$, implying $v_\alpha(b_1)=0$, i.e. $b_1\notin(\pi)$. By the generalization of Fermat's little theorem, $b_1\notin(\pi)\Rightarrow\underline{b_1^{\#\underline{K}-1}}=1$. By assumption $(\#\underline{K}-1)|n(n-1)$, so this implies $\underline{b_1^{n(n-1)}}=1$.

We have $\underline{(1-\gamma)}=\underline{(1-b_1^{n(n-1)})}=\underline{(1-1)}=\underline{0}$. Because $\pi|(1-\gamma)$ we can write $1-\gamma=\pi\cdot c$ for some c. So

$$\operatorname{disc}(\varphi)-\operatorname{disc}(\psi)=\operatorname{disc}(\varphi)\cdot(1-\gamma)=\operatorname{disc}(\varphi)\cdot\pi\cdot c.$$

Therefore, $v_\pi(\operatorname{disc}(\varphi)-\operatorname{disc}(\psi))\ge v_\pi(\operatorname{disc}(\varphi))+1$ and thus $\operatorname{tc}(\operatorname{disc}(\varphi))=\operatorname{tc}(\operatorname{disc}(\psi))$. $\square$

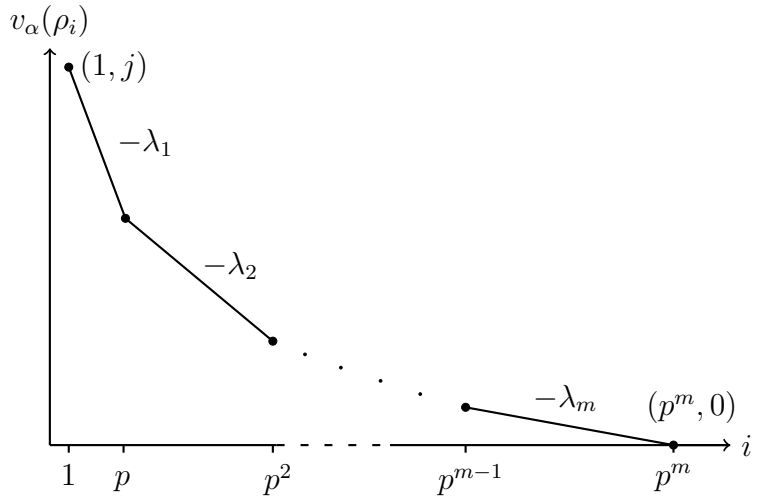

FIGURE 1. Ramification polygon of an Eisenstein polynomial $\varphi \in \mathbb{Q}_p[x]$ of degree p^m with $\operatorname{disc}(\varphi) = p^m + j - 1$ generating a normal cyclic extension, where $\rho(x) = \varphi(\alpha x + \alpha)/(\alpha^{p^m}) = \sum_{i=0}^{p^m} \rho_i x^i \in \mathbb{Q}_p(\alpha)[x]$ with α a root of φ is the ramification polynomial of φ.

5. Proof of Proposition 2.5

Because of Lemma 2.3 we only need to consider the trailing coefficients of the differences of roots in our considerations.

In the proof of the proposition we will use information obtained from the ramification polygon of the polynomial $\varphi \in \mathbb{Q}_p[x]$ under consideration. We recall some of the information that can be obtained from the ramification polygon, see [GP12] for details.

Let α be a root of φ. The ramification polynomial of an Eisenstein polynomial φ of degree n

$$\rho(x) = \varphi(\alpha x + \alpha)/(\alpha^n) = \sum_{i=0}^{n} \rho_i x^i \in \mathbb{Q}_p(\alpha)[x]$$

has the roots $\frac{\alpha^* - \alpha}{\alpha}$ where α^* is a root of φ. The Newton polygon of the ramification polynomial is called the ramification polygon of φ. Its breaks in the can only be at powers of p. The negatives of the slopes λ of the segments are the valuations of the differences of the roots of φ. The length of the segment (in direction of the horizontal axis) with slope λ is the number of roots α^* of φ such that $v_\alpha(\alpha - \alpha^*) = \lambda + 1$.

When $\mathbb{Q}_p[x]/(\varphi)$ is normal the differences of the roots of φ are in $\mathbb{Q}_p[x]/(\varphi)$ and thus the slopes λ_i of the segments of the ramification polygon are integral. Furthermore the roots of the residual polynomials of the segment of slope λ are of the form

$$\underline{\gamma} = \underline{\left(\frac{\alpha^* - \alpha}{\alpha^{\lambda+1}}\right)}$$

and thus $\operatorname{tc}(\alpha^* - \alpha) = \underline{\gamma}$. As $\mathbb{Q}_p[x]/(\varphi)$ is normal we have $\underline{\gamma} \in \mathbb{F}_p$. The normality also implies that the lengths of the segments of the ramification polygon are $p^i - p^{i-1}$ for $1 \leq i \leq m$ and that all elements of $\mathbb{F}_p^\times$ are roots of the residual polynomial of each segment.

Proof of Proposition 2.5. The polynomial φ is an Eisenstein polynomial of degree p^m. As the extension generated by φ is normal, the slopes of the segments are integers. It follows from the symmetry of the roots and the normality of the extension that the breaks in the polygon are exactly at 1, p, p^2, $\ldots$,p^{m-1}, see Figure 1. So the lengths of the segments with finite slope are $p-1$, p^2-p, $\ldots$, p^m-p^{m-1}.

As $\underline{\mathbb{Q}_p^\times} = \mathbb{F}_p^\times$ only has $p-1$ distinct elements it follows from the symmetry of the roots that we get for a fixed root α of φ that:

$$\prod_{v_p(\alpha-\alpha^*)=\lambda_1} \operatorname{tc}(\alpha-\alpha^*) = \prod_{\underline{\gamma}\in\mathbb{F}_p^\times} \underline{\gamma} = \underline{-1} \tag{3}$$

where the α^* are roots of φ and $-\lambda_1$ is the slope of the first segment of the ramification polygon from Figure 1. Similarly, again because of normality and symmetry, taking into consideration the lengths of the segment for the second to the n-th segments with slopes λ_i of length p^i-p^{i-1} using that p is odd, we get:

$$\prod_{v_p(\alpha-\alpha^*)=\lambda_i} \operatorname{tc}(\alpha-\alpha^*) = \left(\prod_{\underline{\gamma}\in\mathbb{F}_p^\times} \underline{\gamma}\right)^{p^{i-1}} = (\underline{-1})^{p^{i-1}} = \underline{-1}. \tag{4}$$

Equations (3) and (4) yield:

$$\begin{aligned}
\operatorname{tc}(\operatorname{disc}(\varphi)) &= (\underline{-1})^{\frac{p^m(p^m-1)}{2}} \prod_{\alpha^*\neq\alpha} \operatorname{tc}(\alpha-\alpha^*) \\
&= (\underline{-1})^{\frac{p^m(p^m-1)}{2}} \prod_{\alpha} \prod_{i=1}^{m} \prod_{v_p(\alpha-\alpha^*)=\lambda_i} \operatorname{tc}(\alpha-\alpha^*) \\
&= (\underline{-1})^{\frac{p^m(p^m-1)}{2}} ((\underline{-1})^m)^{p^m} = (\underline{-1})^{m+\frac{p^m(p^m-1)}{2}}.
\end{aligned}$$

Recall that p is odd. When m is even we have $p^m \equiv 1 \bmod 4$ so $\frac{p^m-1}{2} \equiv 0 \bmod 4$ which implies $\operatorname{tc}(\operatorname{disc}(\varphi)) = 1$. Similarily, when m is odd and $p \equiv 1 \bmod 4$ then $\frac{p^m-1}{2} \equiv 0 \bmod 4$, so $\operatorname{tc}(\operatorname{disc}(\varphi)) = -1$ and when m is odd and $p \equiv 3 \bmod 4$ then $\frac{p^m-1}{2} \equiv 1 \bmod 4$, so $\operatorname{tc}(\operatorname{disc}(\varphi)) = 1$. □

6. Proof of Theorem 2.6 and Corollary 2.7

Before we get to the proof of the theorem and corollary we give some auxiliary results. In the proofs below we use that $\operatorname{tc}(\operatorname{Res}(\varphi,\varphi')) = \operatorname{tc}(\operatorname{disc}(\varphi))\,(\underline{-1})^{\frac{n(n-1)}{2}}$ where $\operatorname{Res}(\varphi,\varphi')$ denotes the resultant of φ and φ' and the degree of φ is n.

To determine the trailing coefficient of the discriminant of Eisenstein polynomials φ of degree p over $\mathbb{Q}_p$ for p odd we distinguish the three cases presented in Table 1. The case $v_p\operatorname{disc}(\varphi) = 2p-2$ in which $\operatorname{Gal}\varphi \cong C_p$ is covered by Proposition 2.5.

Lemma 6.1. Let p be an odd prime, and $\varphi, \psi \in \mathbb{Q}_p[x]$ be of degree p, with $v_p(\operatorname{disc}(\varphi)) = 2p-1$. Then $\operatorname{tc}(\operatorname{disc}(\varphi)) = (\underline{-1})^{\frac{p(p-1)}{2}}$.

Proof. If φ is a degree p polynomial with $v_p(\operatorname{disc}(\varphi)) = 2p-1$, it must be of the form $\varphi(x) = x^p + p(1+ap)$ for $a \in \{1,\ldots,p-1\}$, or generate an isomorphic extension to such a polynomial [JR06, Table 2.1]. However, using Theorem 2.4, we can reduce to the case

TABLE 1. Families of generating polynomials of extensions of degree p of $\mathbb{Q}_p$ for p odd with their Galois group. We have $d = \frac{p-1}{\gcd(\frac{p-1}{m}, j)}$ where m is the order of $\underline{aj}$ in $\mathbb{F}_p^\times$ for $\underline{a} = \underline{\gamma} \cdot (\underline{-1})^{j+1}\underline{j}^{-1}$. See [JR06].

$\varphi \in \mathbb{Q}_p[x]$	Parameters	$v_p(\mathrm{disc}(\varphi))$	$\mathrm{Gal}(\varphi)$
$x^p + apx^j + p$	$1 \le a \le p-1$ $1 \le j \le p-1$ $(j,a) \neq (p-1, p-1)$	$p + j - 1$	$C_p \rtimes C_d$
$x^p - px^{p-1} + p(1+ap)$	$0 \le a \le p-1$	$2p-2$	C_p
$x^p + p(1+ap)$	$0 \le a \le p-1$	$2p-1$	$C_p \rtimes C_{p-1}$

where the polynomials are exactly of this form since we are only concerned with the trailing coefficient of the discriminant. We compute $\varphi'(x) = px^{p-1}$. Then 0 is a root of φ' with multiplicity $p-1$. To show $\mathrm{tc}\,(\mathrm{disc}(\varphi)) = (\underline{-1})^{\frac{p(p-1)}{2}}$ we show $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = \underline{1}$. We have

$$\mathrm{Res}(\varphi, \varphi') = p^p(p(1+ap))^{p-1} = p^{2p-1}(1+ap)^{p-1}.$$

Thus

$$\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = \underline{(1+ap)}^{p-1} = \underline{1}^{p-1} = \underline{1}. \qquad \square$$

Lemma 6.2. Let φ be an Eisenstein polynomial in $\mathbb{Q}_p$ of the form $x^p + apx^j + p$ for $a, j \in \{1, \dots, p-1\}$ and j and a not both equal to $p-1$. Then $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = (\underline{-1})^{j+1}aj$.

Proof. As $\varphi(x) = x^p + apx^j + p$ we have $\varphi'(x) = px^{p-1} + apjx^{j-1} = px^{j-1}(x^{p-j} + aj)$. The polynomial φ' has roots 0 with multiplicity $j-1$, and $r_1, \dots, r_{p-j}$ with $r_k^{p-j} = -aj$ for $1 \le k \le p-j$. Let $\xi \in \overline{\mathbb{Q}_p}$ be a primitive $(p-j)$th root of unity. Then $r_k = \xi^k(-aj)^{\frac{1}{p-j}}$ for $k = 1, \dots, p-j$. Write $\varphi(x) = x^j(x^{p-j} + ap) + p$. Evaluating φ at the roots of φ' we obtain:

$$\mathrm{Res}(\varphi, \varphi') = p^p p^{j-1} \prod_{k=1}^{p-j} \left[\left(\xi^k(-aj)^{\frac{1}{p-j}} \right)^j ((-aj) + ap) + p \right]$$

Hence

$$\begin{aligned}
\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) &= \prod_{k=1}^{p-j} \left[\left(\underline{\xi}^k (\underline{-aj})^{\frac{1}{p-j}} \right)^j \left((\underline{-aj}) + \underline{ap} \right) + \underline{p} \right] \\
&= \prod_{k=1}^{p-j} \underline{\xi}^{kj} (\underline{-aj})^{\frac{j}{p-j}} (\underline{-aj}) = \prod_{k=1}^{p-j} \underline{\xi}^{kj} \cdot \prod_{k=1}^{p-j} (\underline{-aj})^{\frac{p}{p-j}} \\
&= (\underline{-aj})^p \prod_{k=1}^{p-j} \underline{\xi}^{kj} = \underline{-aj} \cdot \underline{\xi}^{j \sum_{k=1}^{p-j} k} \\
&= \underline{-aj} \cdot \underline{\xi}^{j \frac{(p-j)(p-j-1)}{2}} = \underline{-aj} \cdot (\underline{\xi^{\frac{p-j}{2}}})^{j(p-j-1)} \\
&= (\underline{-1})^{j(p-j-1)+1} \underline{aj} = (\underline{-1})^{j+1} \underline{aj}
\end{aligned}$$

$\square$

Proof of Theorem 2.6. Let φ satisfy the above conditions. So we are only considering extensions with non-cyclic Galois group where the valuation of the discriminant is not equal to

$2p-1$. For each of these extensions, there is exactly one polynomial of the form x^p+apx^j+p for $a, j \in \{1, \dots, p-1\}$ where $p+j-1$ is the valuation of the discriminant and j and a are both not equal to $p-1$ [JR06, Proposition 2.3.1].

Thus there exists some $\psi(x) = x^p+apx^j+p$ (for fixed a and j) that generates an extension isomorphic to φ. By Theorem 2.4, $v_p(\mathrm{Res}(\varphi, \varphi')) = v_p(\mathrm{Res}(\psi, \psi'))$ and $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = \mathrm{tc}\,(\mathrm{Res}(\psi, \psi'))$. With Table 1 and Lemma 6.2 we get

$$j = v_p(\mathrm{Res}(\varphi, \varphi')) - p + 1 \text{ and } a = \mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = (\underline{-1})^{j+1}\underline{j}^{-1}.$$

No two distinct $j \in \{1, \dots, p-1\}$ have the same multiplicative inverse modulo p. Also for a fixed j, no two distinct possible values of $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi'))$ give the same value of a. Thus $v_p(\mathrm{Res}(\varphi, \varphi'))$ and $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi'))$ uniquely determine ψ, and therefore the extension. □

Proof of Corollary 2.7. If $v_p(\mathrm{Res}(\varphi, \varphi')) = 2p-1$, then $\mathrm{Gal}(\varphi) = C_p \rtimes C_{p-1}$.

Suppose $v_p(\mathrm{Res}(\varphi, \varphi')) = 2p-2$ and $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = \underline{-1}$. Then φ is either in the first or second family in Table 1, since $v_p(\mathrm{Res}(\varphi, \varphi')) = 2p-2$. By Lemma 6.2 we have $\mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = (\underline{-1})^{j+1}\underline{aj}$. Since $j = p-1$, $\underline{-1} = \mathrm{tc}\,(\mathrm{Res}(\varphi, \varphi')) = (\underline{-1})^p\underline{a(p-1)} = \underline{a}$. Hence φ must be in the second row of the table, that is, $\mathrm{Gal}(\varphi) = C_p$.

Otherwise, compute a and j as in Theorem 2.6. In Table 1, $d = (p-1)/\gcd(\frac{p-1}{m}, j)$ where m is the order of aj in $\mathbb{F}_p^\times$ [JR06]. The size of the Galois group is $p \cdot d$ and the Galois group is $C_p \rtimes C_d$. □

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