Some basics on Coxeter groups and cross ratio on the boundary

Xiangdong Xie
Department of Mathematics and Statistics
Bowling Green State University

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University of North Carolina, Greensboro
The plan

1. Cross ratio on the boundary;
2. Tits representation;
3. Deletion condition and exchange condition;
4. Longest element in a finite Coxeter group
Cross ratio on metric spaces

Classical cross ratio: for $a, b, c, d \in \mathbb{C} \cup \{\infty\}$,

$$cr(a, b, c, d) = \frac{(a - d)(b - c)}{(a - c)(b - d)}.$$ 

Let $X$ be a metric space, and $a, b, c, d \in X$, define

$$cr(a, b, c, d) = \frac{d(a, d)d(b, c)}{d(a, c)d(b, d)}.$$
Metric on the boundary of a tree

Let $T$ be a tree, and $p \in T$. Define a metric on $\partial T$ as follows: for $a \neq b \in \partial T$,
\[ d_p(a, b) = e^{-d(p, ab)}, \]
where $d(p, ab)$ is the distance from $p$ to the geodesic $ab$.

It is easy to see that $d_p$ satisfies the triangle inequality. In fact, $d_p$ is an ultra metric:
\[ d_p(a, c) \leq \max\{d_p(a, b), d_p(b, c)\}. \]

For $a, b, c, d \in \partial T$ we have
\[ cr(a, b, c, d) = \frac{e^{-d(p, ad)} e^{-d(p, bc)}}{e^{-d(p, ac)} e^{-d(p, bd)}} = e^{d(p, ac) + d(p, bd) - d(p, ad) - d(p, bc)}. \]
Cross ratio on the boundary of a tree

The cross ratio (cross difference) on $\partial T$:

$$(a, b, c, d) = d(p, ac) + d(p, bd) − d(p, ad) − d(p, bc).$$

Exercise: $(a, b, c, d)$ is independent of $p$, and is the signed distance from $m(a, b, c)$ to $m(a, b, d)$, with $+$ sign if the direction from $m(a, b, c)$ to $m(a, b, d)$ is the same direction as from $a$ to $b$, and $-$ sign otherwise.

Theorem. A metric tree $T$ with no vertex of valence one is determined up to isometry by the cross ratio on $\partial T$. 
Trees associated with Euclidean buildings

Let $\Delta$ be a locally finite thick Euclidean building. Let $Y$ be a wall in $\Delta$. A subset $Y'$ is said to be parallel to $Y$ if there is some $c \geq 0$ such that $d(y, Y') = c = d(y', Y)$, $\forall y \in Y$, $\forall y' \in Y'$.

Let $P_Y$ be the union of all the sets parallel to $Y$. A basic fact in $\text{CAT}(0)$ space is that $P_Y$ splits isometrically as a product $P_Y = Y \times Z$ for some convex subset of $\Delta$. Due to the dimension consideration it is easy to see that $Z$ is a tree.

By the above discussion, the tree $Z$ can be recovered from the cross ratio. This idea can be used to classify some Euclidean buildings.
Tits representation

Let \((W, S)\) be a Coxeter system with Coxeter matrix \(M = (m_{ij})\). An injective homomorphism \(W \rightarrow GL_n(\mathbb{R})\) (where \(n = |S|\)) due to Tits is constructed as follows.

Write \(S = \{s_1, \cdots, s_n\}\). Fix a basis \((e_i)\), \(1 \leq i \leq n\), for \(\mathbb{R}^n\). Let \(B\) be the symmetric bilinear form determined by

\[
B(e_i, e_j) = -\cos \frac{\pi}{m_{ij}}.
\]

Note \(B(e_i, e_i) = 1\) and \(B(e_i, e_j) \leq 0\) for \(i \neq j\).

Let \(H_i\) be the hyperplane \(H_i = \{v \in \mathbb{R}^n : B(v, e_i) = 0\}\). For each \(i\), define \(\sigma_i : \mathbb{R}^n \rightarrow \mathbb{R}^n\) by:

\[
\sigma_i(v) = v - 2B(e_i, v)e_i.
\]
It is easy to check that $\sigma_i(e_i) = -e_i$ and fixes all points in $H_i$. So $\sigma_i^2 = id$. It can also be checked that $\sigma_i \sigma_j$ has order $m_{ij}$ for $i \neq j$. Hence the map $s_i \rightarrow \sigma_i$ defines a group homomorphism

$$\rho : W \rightarrow GL_n(\mathbb{R}).$$

The map $\rho$ is in fact injective. Hence all Coxeter groups are linear.

**Selberg’s lemma** Finitely generated linear groups are virtually torsion free (have torsion free subgroups of finite index).

**Malcev’s Theorem** Finitely generated linear groups are residually finite.
Let $W$ be a group generated by a finite set of order 2 elements $S \subset W$. Then the following are equivalent:

1. $(W, S)$ is Coxeter system;
2. The deletion condition holds: if $s_1 s_2 \cdots s_k$ is NOT a reduced word in $S$, then there are $i < j$ such that 
   $$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k,$$
   where $\hat{s}_i$ means $s_i$ is removed.
3. The exchange condition holds: if $s_1 \cdots s_k$ is a reduced word in $S$ and $s \in S$, then either $l(sw) = k + 1$ or there is some $i$ such that $w = ss_1 \cdots \hat{s}_i \cdots s_k$. 
Let $(W, S)$ be a finite Coxeter group. Then:
1. There is unique element $w_0$ with the maximal length;
2. Every reduced word in $S$ arises as the initial word for $w_0$; that is, for any $w \in W$, there is some $w' \in W$ satisfying: $l(w) + l(w') = l(w_0)$ and $ww' = w_0$;
3. $w_0^2 = 1$ and $w_0Sw_0 = S$. 