Low dimensional Euclidean buildings: III

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Long time ago, in a building far away
Finite projective plane

A building of type $A_2$ is called a **projective planes**. It's a graph of diameter 3 and girth 6 with two type of vertices called **points** or **lines**.

- If it is finite, every vertex has the same number of neighbor, $q + 1$ (with $q \geq 2$ if thick).
- A projective plane has $q^2 + q + 1$ vertices of each types (points or lines).
- A projective plane has $(q + 1)(q^2 + q + 1)$ edges (chambers).
The game: Dobble
The game: Dobble
Triangle Lattices

Let $\Delta$ be a thick locally finite building of type $\tilde{A}_2$, shortly a **triangle building**.

- Let $q \geq 2$ denote the regularity parameter of $\Delta$.
- Let $I = \{0, 1, 2\}$ denote the types and $V_i$ the set of residues of type $\{j, k\}$ where $\{i, j, k\} = \{0, 1, 2\}$.
- In other words, $V_i$ is the set of vertices of type $i$.
- The residues are finite projective plane of order $q$ (equivalently a finite thick $A_2$-building).
Triangle Lattices

Let $\Gamma$ be a group acting on $\Delta$. Assume the action is:

- **type-rotating**: either $g \in G$ fixes all types or permutes them cyclically.
- **simply-transitive** on the set of vertices on $V = V_0 \cup V_1 \cup V_2$: for every $v, w \in V$ there is a unique $g$ mapping $v$ to $w$.
  - The elements of $G$ are in bijection with the vertices. (Think of $\mathbb{Z}^n$ acting on itself by translation).
Triangle Lattices

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**Theorem (CMSZ)**

*Any such action gives a point-line correspondence and a compatible a triangular presentation. Conversely, any point-line correspondence in a projective plane admitting a triangular presentation yields a triangle building and a lattice as above.*
Triangle Lattices

Generators and Relations

Let $\Gamma$ act simply transitively on the set $V$ of vertices of a building $\Delta$. Fix some $v_0 \in V$, and let $N$ denote the set $\{v \in V : d_V(v_0, v) = 1\}$ of nearest neighbors of $v_0$, i.e. the vertex set of the residue of $v_0$. For each $v \in N$, there must be a unique $g_v \in \Gamma$ such that $g_v v_0 = v$. If $v \in N$, then

$$d_V(g_v^{-1} v_0, v_0) = d_V(v_0, g_v v_0) = 1,$$

and so $g_v^{-1} v_0 \in N$. Write $g_v^{-1} v_0 = \lambda(v)$. Then $g_v^{-1} v_0 = g_{\lambda(v)} v_0$, so that $g_{\lambda(v)} = g_v^{-1}$ for each $v \in N$.

Note that $g_{\lambda(\lambda(v))} = g_{\lambda(v)}^{-1} = g_v$, so that $\lambda : N \rightarrow N$ is an involution. Suppose that $u, v \in N$ and that $d_V(\lambda(u), v) = 1$. Then

$$d_V(v_0, g_u g_v v_0) = d_V(g_u^{-1} v_0, g_v v_0) = d_V(\lambda(u), v) = 1.$$

Thus $g_u g_v v_0 \in N$. Write $g_u g_v v_0 = \lambda(w) = g_w^{-1} v_0$. Then $g_u g_v = g_w^{-1}$, so that

$$g_u g_v g_w = 1.$$ Conversely, if $g_u g_v g_w = 1$ for some $w \in N$, then reversing the above steps, we see that $d_V(\lambda(u), v) = 1$ must hold. Let $T = \{(u, v, w) \in N^3 : g_u g_v g_w = 1\}$. Then

given $u, v \in N, (u, v, w) \in T$ for some $w \in N$ if and only if $d_V(\lambda(u), v) = 1$. 
Definition 2.1. Let $P$ and $L$ be the sets of points and lines respectively in a projective plane $\Pi$. A bijection $\lambda: P \to L$ is called a **point-line correspondence** in $\Pi$. A subset $\mathcal{T} \subseteq P^3$ is then called a **triangle presentation** compatible with $\lambda$ if the two following conditions hold:

1. For all $x, y \in P$, there exists $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \lambda(x)$ in $\Pi$. In this case, $z$ is unique.

2. If $(x, y, z) \in \mathcal{T}$, then $(y, z, x) \in \mathcal{T}$.

Example 2.2. The projective plane $\text{PG}(2, 2)$ can be defined by $P = L = \mathbb{Z}/7\mathbb{Z}$ with line $x \in L$ being adjacent to the points $x + 1$, $x + 2$ and $x + 4$ in $P$. Consider the point-line correspondence $\lambda: P \to L: x \in P \mapsto x \in L$ in $\Pi$. Then

$$\mathcal{T} := \{(x, x + 1, x + 3), (x + 1, x + 3, x), (x + 3, x, x + 1) \mid x \in P\}$$

is a triangle presentation compatible with $\lambda$. Indeed, (ii) is obviously satisfied and, for $x, y \in P$, it is apparent that there exists (a unique) $z \in P$ such that $(x, y, z) \in \mathcal{T}$ if and only if $y \in \{x + 1, x + 2, x + 4\}$, which is exactly the set of points on the line $\lambda(x)$.
Definition 3.5. Let \( \lambda : P \rightarrow L \) be a point-line correspondence in a projective plane \( \Pi \). A subset \( \mathcal{T} \subseteq P^3 \) is called a triangle partial presentation compatible with \( \lambda \) if the two following conditions hold:

1. For all \( x, y \in P \), if there exists \( z \in P \) such that \( (x, y, z) \in \mathcal{T} \) then \( y \in \lambda(x) \) and \( z \) is unique.
2. If \( (x, y, z) \in \mathcal{T} \), then \( (y, z, x) \in \mathcal{T} \).

We directly have the following.

Lemma 3.6. Let \( \lambda : P \rightarrow L \) be a point-line correspondence in a projective plane \( \Pi \) of order \( q \). A subset \( \mathcal{T} \subseteq P^3 \) is a triangle presentation compatible with \( \lambda \) if and only if it is a triangle partial presentation compatible with \( \lambda \) and \( |\mathcal{T}| = (q + 1)(q^2 + q + 1) \).

Proof. This is clear from the definitions, since there are exactly \((q + 1)(q^2 + q + 1)\) pairs \((x, y) \in P^2\) with \( y \in \lambda(x) \).

We now define the score of a point-line correspondence as follows.

Definition 3.7. Let \( \lambda : P \rightarrow L \) be a point-line correspondence in a projective plane \( \Pi \) of order \( q \). The score \( S(\lambda) \) of \( \lambda \) is the greatest possible size of a triangle partial presentation compatible with \( \lambda \).
3.1 The graph associated to a point-line correspondence

In the context of triangle presentations, it is natural to associate a particular graph to each point-line correspondence \( \lambda: P \to L \) of a projective plane \( \Pi \).

**Definition 3.1.** Let \( \lambda: P \to L \) be a point-line correspondence in a projective plane \( \Pi \). The **graph** \( G_\lambda \) associated to \( \lambda \) is the directed graph with vertex set \( V(G_\lambda) := P \) and edge set \( E(G_\lambda) := \{(x, y) \in P^2 \mid y \in \lambda(x)\} \).

For \( \lambda \), admitting a triangle presentation can now be rephrased as a condition on its associated graph \( G_\lambda \). In order to state this reformulation, we first define what we will call a **triangle** in a directed graph.

**Definition 3.2.** Let \( G \) be a directed graph. A set \( \{e_1, e_2, e_3\} \) of edges in \( G \) such that the destination vertex of \( e_1 \) (resp. \( e_2 \) and \( e_3 \)) is the origin vertex of \( e_2 \) (resp. \( e_3 \) and \( e_1 \)) is called a **triangle**. If two of the three edges \( e_1, e_2 \) and \( e_3 \) are equal, then they are all equal. In this case, the triangle contains only one edge and is also called a **loop**.

**Lemma 3.4.** Let \( \lambda: P \to L \) be a point-line correspondence in a projective plane \( \Pi \). There exists a triangle presentation compatible with \( \lambda \) if and only if there exists a partition of the set of edges \( E(G_\lambda) \) of \( G_\lambda \) into triangles.

**Proof.** Via the above bijection, a partition of \( E(G_\lambda) \) into triangles exactly corresponds to a triangle presentation compatible with \( \lambda \). \( \square \)
3.3 Scores of correlations

When \( \lambda: P \rightarrow L, \ L \rightarrow P \) is a correlation of a (self-dual) projective plane \( \Pi \) of order \( q \), i.e. a map such that \( \lambda(p) \ni \lambda(\ell) \) if and only if \( p \in \ell \), there is an explicit formula for the score of the point-line correspondence \( \lambda: P \rightarrow L \).

**Proposition 3.10.** Let \( \lambda: P \rightarrow L, \ L \rightarrow P \) be a correlation in a projective plane \( \Pi \) of order \( q \). Let \( a(\lambda) \) be the number of points \( p \in P \) such that \( \lambda^3(p) \ni p \) and let \( b(\lambda) \) be the number of points \( p \in P \) such that \( \lambda^3(p) \ni p \) and \( \lambda^6(p) = p \). Then

\[
S(\lambda) = (q + 1)(q^2 + q + 1) - (2q - 3) \cdot a(\lambda) - b(\lambda).
\]

**Theorem 3.11** (Devillers–Parkinson–Van Maldeghem). Let \( \lambda: P \rightarrow L, \ L \rightarrow P \) be a correlation in a finite projective plane \( \Pi \). Then there exists \( p \in P \) such that \( p \in \lambda(p) \).
## Score of a Correlation

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<tr>
<th># of concerned $\lambda$</th>
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<th>$b(\lambda)$</th>
<th>$S(\lambda)$</th>
<th>$s(\lambda)$ (mean)</th>
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Table 3.1: Scores of the correlations of the Hughes plane of order 9.
While there exists $e \in E(G_\lambda)$ such that there is a unique triangle $t$ in $G_\lambda$ containing $e$, choose this triangle $t$, remove the edge(s) of $t$ from $G_\lambda$ and start again this procedure. If, at the end, there is no more triangles in $G_\lambda$, then we say that the score-algorithm succeeds and that the estimated score $s(\lambda)$ of $\lambda$ is the number of edges that are covered by the chosen triangles. Otherwise, there still are triangles in $G_\lambda$ but all edges are contained in 0 or at least 2 triangles. In this case, we say that the score-algorithm fails. For a pseudo-code, see Algorithm 1.
Algorithm 1: Computing the estimated score $s(\lambda)$ of $\lambda$

1. $score \leftarrow 0$
2. $edgesInOneTriangle \leftarrow true$
3. while $edgesInOneTriangle = true$ do
4.     $edgesInOneTriangle \leftarrow false$
5.     for $e$ in $E(G_\lambda)$ do
6.         if $e$ is contained in exactly one triangle $t$ of $G_\lambda$ then
7.             $edgesInOneTriangle \leftarrow true$
8.             remove the edge(s) of $t$ from $E(G_\lambda)$
9.             if $t$ is a loop then
10.                $score \leftarrow score + 1$
11.            else
12.                $score \leftarrow score + 3$
13.         end if
14.     end if
15. end while
16. if there still are triangles in $G_\lambda$ then
17.     return $FAIL$
18. end if
19. return $score$
Improving the Score: Algorithm 2

Lemma 3.15. Let $\lambda: P \to L$ be a point-line correspondence in a projective plane $\Pi$ of order $q$ and let $a, b \in P$. Define $\lambda_{a,b}: P \to L$ by $\lambda_{a,b}(x) := \lambda(x)$ for all $x \in P \setminus \{a, b\}$, $\lambda_{a,b}(a) := \lambda(b)$ and $\lambda_{a,b}(b) := \lambda(a)$. Then $|S(\lambda_{a,b}) - S(\lambda)| \leq 6(q + 1)$.

**Algorithm 2:** Finding a point-line correspondence $\lambda$ with $s(\lambda) = 910$

1. $\lambda \leftarrow$ some correlation of the Hughes plane;
2. **while** $s(\lambda) < 910$ **do**
   3. $\text{visited}[\lambda] \leftarrow$ true;
   4. $\text{bestA} \leftarrow -1; \text{bestB} \leftarrow -1;
   5. $\text{bestScore} \leftarrow -1;$
   6. **for** $a$ in $P$ **and** $b$ in $P$ **do**
   7. **if** $\text{visited}[\lambda_{a,b}] = \text{false and } s(\lambda_{a,b}) > \text{bestScore}$ **then**
   8. $\text{bestScore} \leftarrow s(\lambda_{a,b});$
   9. $\text{bestA} \leftarrow a;$
   10. $\text{bestB} \leftarrow b;$
   11. $\lambda \leftarrow \lambda_{\text{bestA}, \text{bestB}};$
3. **return** $\lambda;$
\begin{table}[h]
\centering
\begin{tabular}{|c|cccccccccc|}
\hline
$\lambda$ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
0. & 20 & 0 & 44 & 75 & 78 & 77 & 50 & 76 & 37 & 3 \\
1. & 54 & 39 & 30 & 8 & 88 & 68 & 18 & 34 & 65 & 57 \\
2. & 70 & 82 & 42 & 23 & 38 & 90 & 81 & 13 & 61 & 69 \\
3. & 73 & 4 & 83 & 22 & 58 & 28 & 59 & 55 & 64 & 60 \\
4. & 56 & 2 & 87 & 84 & 26 & 45 & 53 & 11 & 80 & 41 \\
5. & 25 & 14 & 63 & 72 & 7 & 32 & 62 & 86 & 51 & 46 \\
6. & 36 & 27 & 31 & 29 & 79 & 33 & 16 & 71 & 85 & 24 \\
7. & 89 & 35 & 17 & 19 & 5 & 47 & 67 & 10 & 66 & 43 \\
8. & 6 & 21 & 1 & 52 & 74 & 40 & 12 & 48 & 9 & 15 \\
9. & 49 & & & & & & & & & \\
\hline
\end{tabular}
\caption{Table B.2: Triangle presentation $T$ compatible with $\lambda$.}
\end{table}
## Results

### A The Hughes plane of order 9

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Radu’s C++ program

A few things to know about the C++ code:

- Radu uses the fact that the lines 0, 1, 10 and 30 generate the Hughes plane.
- It was too slow to check all pairs $a, b$, so he tests and selects only a few pairs. Especially the vertices for which few triangles have been used. He calls them **bad** vertices.

What the code does:

- Generates correlations until it finds one with a good score $\geq 750$.
- Apply the improving algorithm, which permutes some $a$ and $b$ to see if it gets to the score max of 910. (Keeps track of the permutations to not fall in a local maximum).
- If after 150 steps the score is still low, it moves on to the next correlation.
Finite projective plane $A_2(\mathbb{F}_2)$. 
Goal 1: Radu’s lattice

- Finite projective plane $A_2(F_2)$. 

![Diagram of a finite projective plane $A_2(F_2)$]
Goal 1: Radu’s lattice

- A point-line correspondence $\lambda$ forming pairs.
Goal 1: Radu’s lattice

- The incidence relation: \( \text{point} \subset \text{line} \)
Goal 1: Radu’s lattice

- A graph $G_{\lambda}$ associated to the point line correspondence $\lambda$. 

![Graph](image-url)
Goal 1: Radu’s lattice

- The triangle presentation $\mathcal{T}$ is a cover of $G_\lambda$ by disjoint triangles.
Goal 1: Radu’s lattice

- Triangle can also mean loop.
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 

![Diagram of Radu's lattice]
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 
Goal 1: Radu’s lattice

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![Diagram of Radu's lattice](image-url)
Goal 1: Radu’s lattice

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Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 

![Diagram of a lattice with arrows connecting points]
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$.牛
Goal 1: Radu’s lattice

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Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $T$. 
Goal 1: Radu’s lattice

- So we remove triangles (or loop) one by one to obtain $\mathcal{T}$. 
Goal 1: Radu’s lattice

▶ No triangle left, so we the triangle we removed form a cover of $G_\lambda$. Pretty lucky!