ON THE EXPONENTIAL FUNCTION

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Abstract. The natural exponential function is one of the most important functions students should learn in calculus classes. The applications range from mathematics, statistics, natural sciences, and economics. Despite its wide use and importance, instructors often struggle with the proper definition. The purpose of this note is to provide a short, self-contained exposition on the natural exponential function $\exp(x)$ starting from an accessible definition to the derivation of all of its properties. We will also discuss other common definitions of the exponential function and show its application in natural sciences.

1. Definitions of the exponential function

There are many available definitions of $\exp(x)$. Wikipedia, see [7], lists the following 5 definitions.

(D1) $\exp(x) \stackrel{\text{def}}{=} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

(D2) $\exp(x)$ is the inverse to $\ln x \stackrel{\text{def}}{=} \int_1^x \frac{1}{s}\,\mathrm{d}s$.

(D3) $\exp(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

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(D4) $\exp(x)$ is the only continuous function $f(x)$ satisfying $f(a+b) = f(a)f(b)$ for all $a, b \in \mathbb{R}$ and $f(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

(D5) $\exp(x)$ is the solution of the differential equation $y' = y$ satisfying $y(0) = 1$.

We can choose any definition from the above, and as soon as we choose one, the others become properties of the function. So the question is which definition should we choose.

The definition (D1) contains the most primitive terms. Theoretically, we could introduce the exponential function using this definition as early as possible. The drawback of this definition is that derivation of other properties from it is too technical, see [4, pages 51, and 133].

In calculus classes, the exponential function is usually defined by (D2), see e.g. [6, p. 425], [1, p. 428], [8, p. 331]. This approach requires waiting till the definite integral is introduced.

The definition (D3) is often used in real analysis classes, see [5, p. 178]. This is one of the most universal definitions since it allows us to use the same approach to define other functions (like $\sin x \stackrel{\text{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, or $\cos x \stackrel{\text{def}}{=} \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$). The formula can also be used to define exponentials of matrices (and linear operators in general), see [3]. But since the formula requires the knowledge of series, it is not used in any early classes.

The property $\exp(a+b) = \exp(a)\exp(b)$, which is the core of the definition (D4), is one of the reasons why $\exp(x)$ is an important function. The definition (D4) is probably the most abstract definition of the exponential function. Although probably the most beautiful of all of the definitions, it is "too far" from calculus to be used in calculus at all.

The property $\exp(x)' = \exp(x)$ is the core of definition (D5). It is this property that makes the exponential function important for calculus. It is also the reason why students like to differentiate the exponential function. Although the definition implicitly contains a differential equation and thus seems to be a highly advanced definition, it could be explained to students as soon as they learn what a derivative is.

In the rest of the paper we will advocate why we should choose (D5) for the definition of the exponential function.

2. Definition of $\exp(x)$ as a unique solution of $y' = y, y(0) = 1$

Let us start from the beginning.

Definition. The natural exponential function $\exp(x)$ is defined to be the only function $y = y(x)$ that satisfies the following two conditions

(E1) $y'(x) = y(x)$, for all $x \in (-\infty, \infty)$, and
(E2) $y(0) = 1$.

There is only one problem with the above mathematical definition: there is no guarantee that such a function exists and is unique. However, both the existence and the uniqueness are guaranteed by Picard's theorem [2, p. 110]. The general version of the theorem is quite advanced and this is probably the reason why the above definition is not widely spread in calculus classes. However, it is enough to have the following weaker version.

Theorem 1 (Picard)**.** *For any numbers k and c_0 there exists a unique function $y(x)$ satisfying*

- $y'(x) = ky(x)$, *for all* $x \in (-\infty, \infty)$, *and*
- $y(0) = c_0$.

If we are teaching calculus for application-oriented students, we may use the following example on uninhibited growth which also turns out to be useful in visualizing some of the properties of the exponential function.

Visual example - uninhibited growth. (Compare e.g. [1, p.605]) Assume a cell splits every $T = \ln 2$ into two new cells and that there are originally c_0 cells at time $t = 0$. Then $c_0 \exp(x)$ is the number of cells at time $t = x$.

So now we have the definition of the natural exponential function. It remains to prove its properties.

3. Properties of $\exp(x)$.

Let us start by proving the properties (D1)-(D4) from the beginning of the paper.

Fact 2. $\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

Proof. Denote $y(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. Then we have

$$\begin{aligned}
y'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) \\
&= \lim_{n\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} n\cdot\left(1+\frac{x}{n}\right)^{n-1}\cdot\frac{1}{n} \\
&= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\cdot\left(1+\frac{x}{n}\right)^{-1} \\
&= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-1} = y(x).
\end{aligned}$$

The second equality (where we interchanged the limit and differentiation) follows by [5, Theorem 7.17]. Thus, we have $y'(x) = y(x)$. Moreover,

$$y(0) = \lim_{n\to\infty}\left(1+\frac{0}{n}\right)^n = 1.$$

Hence, by the uniqueness of the exponential function, $y(x) = \exp(x)$ which is exactly what we wanted to prove. □

Fact 3. $\exp(x)$ *is an inverse to* $\ln x \stackrel{def}{=} \int_1^x \frac{1}{s}\, ds$.

Proof. Let $y(x)$ denote the inverse to $\ln x$. By the fundamental theorem of calculus, [1, p. 403],

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}.$$

Thus, by the theorem on differentiation of inverse functions, [1, p. 249],

$$y'(x) = \frac{1}{\frac{1}{y(x)}} = y(x).$$

Because $\ln 1 = 0$ we get $y(0) = 1$ and thus, by the uniqueness of the exponential function, $y(x) = \exp(x)$ which we wanted to prove. □

Fact 4. $\exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Proof. Denote $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then we have

$$\begin{aligned} y'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} n \cdot \frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = y(x) \end{aligned}$$

where the second equality holds because of the theorem on differentiation of the power series, [1, p. 704]. Thus, $y'(x) = y(x)$; and since $y(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$, we must have, by the uniqueness of the exponential function, $y(x) = \exp(x)$. □

Fact 5. $\exp(a + b) = \exp(a)\exp(b)$ *for all* $a, b \in (-\infty, \infty)$.

Proof. Instead of a rigorous proof, we will demonstrate this fact using the example on an uninhibited growth of cells.

Consider the following "experiment". First, let a colony of initially c_0 cells grow for time $t = a$. At time $t = a$, the colony will consist of $c_0 \exp(a)$ cells. Next, divide the colony into $\exp(a)$ colonies, each one of them consisting of c_0 cells. Finally, let all colonies grow for an additional time $t = b$. After that time, each such a colony will consist of $c_0 \exp(b)$ cells. Together, after time $t = a + b$, there will be $c_0 \exp(a)\exp(b)$ cells.

On the other hand, based on what we assumed about the cell growth, the division of the colony at time $t = a$ did not have any effect on individual cells, so after the time $t = a + b$ there are $c_0 \exp(a + b)$ cells. By matching the two numbers we have

$$c_0 \exp(a + b) = c_0 \exp(a)\exp(b),$$

which is exactly what we wanted to prove. □

Remark. For simplicity, we assumed that $\exp(a)$ is an integer, but this was only in order to make the example really visual. We also assumed that the division of the original colony at time $t = a$ did not have any effect on the cell growth. The mathematical counterpart of this idea is the Picard's theorem.

Now we will prove some elementary properties of the natural exponential function.

Fact 6. (i) $\exp(x)$ *is continuous and differentiable for all* x.

(ii) $\exp(x) > 0$ *for all* x.

(iii) $\exp(x)$ *is strictly increasing and concave up.*

(iv) $\exp(x) \geq 1 + x$ *for all* x.

(v) $\lim_{x\to\infty} \exp(x) = \infty$ *and* $\lim_{x\to-\infty} \exp(x) = 0$.

Proof. (i) Since $\exp(x)$ solves an equation $y' = y$, it must be differentiable. Moreover, every differentiable function is continuous (see e.g. [1, p. 184]).

(ii) If there is x such that $\exp(x) \leq 0$, then, by the intermediate value theorem, [1, p. 149], there must be x' such that $\exp(x') = 0$ (because $\exp(0) = 1 > 0$). It follows

$$\begin{aligned} 1 &= \exp(0) = \exp(x' + (-x')) \overset{\text{Fact 5}}{=} \exp(x')\exp(-x') \\ &= 0 \cdot \exp(-x') = 0, \end{aligned}$$

which is a contradiction. Hence, there is no x such that $\exp(x) \leq 0$, i.e. $\exp(x) > 0$ for all x.

(iii) Since, by the definition and (ii),

$$(\exp(x))' = \exp(x) > 0,$$

$\exp(x)$ is strictly increasing. Since, again by the definition and (ii),

$$(\exp(x))'' = \Big((\exp(x))'\Big)' = \Big(\exp(x)\Big)' = \exp(x) > 0,$$

$\exp(x)$ is concave up.

(iv) The function $1 + x = \exp(0) + \exp'(0)x$ is a tangent line to $\exp(x)$ at $x = 0$. Since, by (iii), $\exp(x)$ is concave up, the graph of $\exp(x)$ must be above the tangent line and the inequality follows.

(v) The first part follows directly from (iv). Indeed,

$$\lim_{x\to\infty} \exp(x) \geq \lim_{x\to\infty} 1 + x = \infty.$$

For the second part, we know that the limit $L = \lim_{x\to-\infty} \exp(x)$ exists (as $\exp(x)$ is increasing) and is at least 0 (by (ii)). Since $\exp(x)$ is its own derivative, we also have $L = \lim_{x\to-\infty} (\exp(x))'$. We claim that this is possible only if $L = 0$.

The proof that $L = 0$ is by contradiction. Indeed, for a contradiction, assume $L > 0$. By the definition of a limit (but without going too much into ε, δ exercise), we can take a number x_0 so that for all $x < x_0$

(a) $L \approx \exp(x)$, and
(b) $L \approx \exp'(x)$.

We will now consider a fraction $\frac{\exp(x)-\exp(y)}{x-y}$ and we will estimate it in two ways.

First, we show that it has to be almost 0. Take $x, y < x_0$ with $|x-y|$ large enough so that $4L/|x-y| \approx 0$. Then,

$$\begin{aligned} 0 &\leq \left|\frac{\exp(x)-\exp(y)}{x-y}\right| \leq \frac{|\exp(x)|+|\exp(y)|}{|x-y|} \\ &\leq \frac{4L}{|x-y|} \approx 0, \end{aligned}$$

i.e. $\frac{\exp(x)-\exp(y)}{x-y} \approx 0$.

On the other hand, we will show that the fraction should be approximately L. By the mean-value theorem, there exists c between x and y (thus $c < x_0$ and consequently $\exp'(c) \approx L$) such that

$$L \approx \exp'(c) = \frac{\exp(x)-\exp(y)}{x-y} \approx 0.$$

It means that L should be arbitrarily close to 0. It is possible only if $L = 0$, contradicting our assumption $L > 0$. □

4. Conclusion

We have defined $\exp(x)$ in a relatively elementary way as a unique solution of the initial value problem

$$\begin{aligned} y'(x) &= y(x), \\ y(0) &= 1 \end{aligned}$$

and showed how easily all properties of the exponential function follow. We have also seen how the exponential function relates to examples from mathematical biology, namely the uninhibited growth of cells.

We will continue the work to show that the functions $\cos x$ and $\sin x$, respectively, can be defined in a very similar manner: as a real part and an imaginary part, respectively, of the unique solution of the initial value problem

$$\begin{aligned} y'(x) &= iy(x) \\ y(0) &= 1. \end{aligned}$$

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