Lecture 8 Exercises

Let $f : \mathbb{R} \to \mathbb{R}$ be a "Schwarz" function. That is, f is infinitely differentiable, and all derivatives decay super-polynomially: for all $n, m \geq 1$,

$$|f^{(n)}(x)| \ll_{n,m} \frac{1}{|x|^m + 1}.$$

(The notation is that the implied constant can depend on n and m.)

(1) Show that the space of Schwarz class functions is a vector space, and contains more than just the zero function. [Hint: try the Gaussian, $f(x) = e^{-\pi x^2}$.]

(2) "Automorphise" f to F by defining

$$F(x) := \sum_{n \in \mathbb{Z}} f(n + x).$$

Show that $F(x + 1) = F(x)$, so F is a function on the circle, $\mathbb{R}/\mathbb{Z}$, and can be represented by its Fourier series,

$$F(x) = \sum_{m \in \mathbb{Z}} \widehat{F}(m) e(mx).$$

Here $e(z) = e^{2\pi i z}$ and

$$\widehat{F}(m) = \int_{\mathbb{R}/\mathbb{Z}} F(x) e(-mx) dx$$

is the Fourier transform on the circle.

(3) Show that

$$\widehat{F}(m) = \widehat{f}(m),$$

where the latter is the Fourier transform on the line,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e(-\xi x) dx.$$

(4) Combing the above to prove the "Poisson summation" formula,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \widehat{f}(m).$$

(5) For $t > 0$, define

$$f_t(x) := f\left(\frac{x}{t}\right),$$

and show that $\widehat{f_t}(\xi) = t\widehat{f}(\xi t)$. Conclude that

$$\sum_{n \in \mathbb{Z}} f\left(\frac{n}{t}\right) = t \sum_{m \in \mathbb{Z}} \widehat{f}(nt). \tag{1}$$

(6) Verify that the Gaussian $f(x) = e^{-\pi x^2}$ is its own Fourier transform, $\widehat{f}(\xi) = e^{-\pi\xi^2}$. Apply (1) to show that

$$\sum_{n\in\mathbb{Z}} e^{-\pi(n/100)^2} = 100 \sum_{m\in\mathbb{Z}} e^{-\pi(100m)^2}.$$

Notice that the terms on the left hand side are extremely small once $|n| > 100$, so it takes effectively ~ 200 terms to determine the value of the sum. How many terms contribute to the sum on the right hand side??

Open Problem: Consider the Markoff equation

$$V \ : \ x^2 + y^2 + z^2 - 3xyz = 0.$$

Observe that $\mathbf{x}_0 := (1, 1, 1) \in V(\mathbb{Z})$. Observe that if x, y are given, then the Markoff equation is quadratic in z; then there are two solutions to this quadratic, z and z', say, and (as in the Apollonian case) $z + z'$ is a simple (but now non-linear!) function of x, y. Work out what this function is. Replacing z by z' is referred to as "Vieta flipping." A lovely theorem of Markoff says that $V(\mathbb{Z}) \setminus (0, 0, 0)$ is generated from $\mathbf{x}_0$ via these Viete flips; that is, the orbit of $\mathbf{x}_0$ under this group of Vieta moves fills out all (non-trivial) solutions over $\mathbb{Z}$.

Question (Strong Approximation): Does this one orbit also fill out all solutions to the equation mod p? That is, is the graph with vertices points of $V(\mathbb{Z}/p)\setminus(0, 0, 0)$ and edges connecting points by Viete moves, connected? [This problem was solved recently by Will Chen https://arxiv.org/abs/2011.12940 building on foundational work of Bourgain-Gamburd-Sarnak https://arxiv.org/pdf/1607.01530.pdf.]

Open Problem (Super Approximation): Is this family of graphs an Expander???!!!