Zigzag Construction of expanders - exercises

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1 Zigzag Product

The zigzag product graph was described as a three-step-walk, first in H, then in G, then again in H, $\tilde{M}_H\tilde{M}_G\tilde{M}_H$. What happens if we only take two steps (one in G and one in H). Would the graph still be an expander?

2 Expander Mixing Lemma

Let $G = (V, E)$ be a d-regular graph on n vertices. Prove that for every $S, T \subseteq V$:

$$|E(S,T) - \frac{d|S||T|}{n}| \leq \lambda d\sqrt{|S||T|}$$

where $E(S,T)$ denotes the number of edges with one endpoint in S and one in T.

3 Replacement Product

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs such that $|V_G| = n$ and G is D-regular, and such that $|V_H| = D$ and H is d-regular. Suppose that for each vertex of G the edges adjacent to it are ordered arbitrarily. The replacement product is the graph $\Gamma = (V, E)$ defined by

$$V = V_G \times V_H \qquad E = E^{out} \cup E^{in}$$

where

$$E^{in} = \{\{(v,i),(v,j)\} \mid \{i,j\} \in E_H\}$$

and

$$E^{out} = \{\{(v,i),(u,j)\} \mid \ v \text{ is the } j\text{th neighbor of } u, \text{ and } u \text{ is the } i\text{th neighbor of } v.\}$$

- Prove that the replacement product of two expanders is an expander. Can you modify the definition to improve upon the Cheeger constant?
- Give an explicit construction of an infinite sequence of 3-regular graphs with a uniform lower-bound on their Cheeger constant.