Lecture 1

Expanders: combinatorial
  definition

Slogan: An expander graph is a finite graph, which is highly connected (in a robust way) sparse (not too many edges)

Remarkable facts:
(1) Such graphs exist
(2) They have amazing applications [in TCS, geometry, number theory, combinatorics, knot theory, arithmetic geometry ...]
1.1 - Graphs, examples

Informally:
- vertices
- edges between some pairs of vertices [unoriented]

Often we work with finite graphs \((\leq V, E)\) both finite; we write \(|\Gamma| = |V|\) for a graph

The number of vertices.

A graph is \(d\)-regular if
all vertices have d neighbors (with multiplicity) [loop counts as 1] not regular

Examples:

* $C_m$ (cycles)

(2-regular)

(1-regular) $C_1$

$C_2$ (2-regular)

Def. (Girth)

A graph $\Gamma$ has girth equal to $m \geq 1$ if it contains an $m$-cycle, and no smaller one.
Def. (Forest)
A graph is a forest if its girth is infinite.

Def. (Bipartite)
\( \Gamma \) is bipartite if
\[ V = V_0 \sqcup V_1 \]

with all edges joining \( V_0 \)
to \( V_1 \).
Ex. \( K_m \): complete graph on \( m \) vertices: one edge between any vertices \( x \neq y \)

\[ ((m-1)\text{-regular}) \]

1.2. Distance

**Def.** \( G = (U, E) \) graph

\[ x, y \in V \]

\[ d_G(x, y) = \min \{ \text{length}(y) \mid y \text{ is a path in } G \} \]
Prop. \( d \) is a distance on \( X \) if
\[
\begin{aligned}
&d(x, y) = 0 \iff x = y \\
&d(x, y) = d(y, x) \\
&d(x, z) \leq d(x, y) + d(y, z)
\end{aligned}
\]
→ notions from geometry
  - connectedness: if the distance is always finite
  - diameter:

\[
\text{diam}(\Gamma) = \max_{(x,y) \in U^2} d(x, y)
\]

For many applications, having a "small" diameter is a good thing.

Ex. \[\text{diam}(C_m) \approx \frac{m}{2} \left( = \frac{1}{2} \right)\]

- \[\text{diam}(K_m) = 1\]
Lemma. If $(\Gamma_n)$ is a sequence of finite $d$-regular graphs ($d$ fixed), then

$$\exists c > 0, \quad \text{diam} (\Gamma_n) \geq c \log (|\Gamma_n|)$$

$\Rightarrow$ The best diameter bound, without increasing the number of neighbors ("valencies") is about $\log (|\Gamma|)$.

1.3. Cayley graphs

Def. Let $G$ be a group.

Let $S \subseteq G$ be a subset (not nec. a subgroup), with...
$S=S^{-1}$. The Cayley graph of $G$ relative to $S$, denoted $\mathcal{G}(G, S)$ is the graph with

- vertices $= G$
- edges join $g$ to $gs$ for $g \in G$ and some $s \in S$

Example: $G = \mathbb{Z}/m\mathbb{Z}$

$S = \{1, -1\}$

$\mathcal{G}(\mathbb{Z}/m\mathbb{Z}, \{1, -1\})$

$C_m$
Changing $S$ changes the graph!

$G = \mathbb{F}_2$, free group on two generators

$S = \{ a, a^{-1}, b, b^{-1} \}$

$\mathbb{Z}^2$

$(\pm 1,0), (0, \pm 1)$

grid

$\rightarrow$ Infinite 4-regular tree

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Facts:

1. $G(G,S)$ is $|S|$-regular
2. $G(G,S)$ is connected

\[ S \text{ generates } G \]
(3) $G(6, 5)$ is bipartite

[Exercise 1.9]

$$\exists \, \varepsilon : G \rightarrow \{ \pm 1 \} \text{ group morphism, surjective, with } \varepsilon(s) = -1 \text{ for all } s \in S$$

[Counterex. $G = S_5$, $S = \{ \tau, \tau_2 \}$, $\tau, \tau_2 \in A_5$, $\ldots$]

1.4. Expansion in graphs

Want to define "sparse and robustly connected" sequences of graphs.

Sparse: every vertex has a bounded number of neighbors uniformly.
Connectedness: we ask for a small diameter.

Not so good:

\[ \text{diameter: } \leq 3 \text{ but removing a single edge disconnects the graph!} \]

To get a better definition, one defines an invariant that "measures" robustness.
Definition (Cheeger constant)

\[ h(G) = \min_{\emptyset \neq W, \frac{|W|}{|V|} \leq \frac{1}{2}} \left\{ \frac{|\partial W|}{|W|} \right\} \]

where \( \partial W \) is the set of edges with one extremity in \( W \), the other outside \( W \).

\[ h(G) \] "small" means that \( G \) can be disconnected "easily."
Lemma. \( h(\Gamma) > 0 \) \[ \uparrow \] \( \Gamma \) is connected.

Def. (Expander graphs)
A sequence \( (\Gamma_n)_{n \geq 1} \) of finite graphs is an expander (family) \( \iff \)

(i) \( \lim_{n \to \infty} |\Gamma_n| = +\infty \)

(ii) max. nb. of neighbors is uniformly bounded (e.g. all \( \Gamma_n \) is \( d \)-regular with \( d \) fixed).
(iii) \( \exists c > 0 \) s.t. \( \forall n \geq 1, \ h(\Gamma_n) \geq c \).

Q. Do these exist??