Zigzag construction of Expander graphs

We construct an infinite sequence of \( d \)-regular expander graphs algorithmically via elementary operations.

Two operations: 1) Powering 2) Zigzag

1) **Powering**: \( G \to G^2 \)

- \( G^2 \) is the graph on the same set of vertices, and edges for two-step-walks. More accurately,
  \[
  A_{G^2} = A_G \cdot A_G. \quad \text{(also} \quad M_{G^2} = M_G \cdot M_G, \quad M_G = \frac{1}{d} A_G)\]

- \( G^2 \) is \( d^2 \)-regular

Claim: \( G^2 \) is \( d^2 \)-regular, has eigenvalues \( (\lambda_{G^2})_{i=1}^{n} \).

(we can remove the \( d \) self loops: \( A_{G^2} - d \cdot I_n \))

2) **Zigzag**: \( G, H \) two graphs \( G \odot H \)

Notation: \( G \) is an \((m, m, \omega)\)-graph denotes \( m \)-regular graph on \( n \) vertices with \( \max(1, \omega) \leq d \).

\[
\forall i \geq 1 \quad \lambda_i \leq \omega \quad 1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n \geq -1
\]

**Theorem [RVW '00]**: Assume \( G \) is \((m, m, \omega)\)-graph and \( H \) is \((m, d, \beta)\)-graph. Then \( G \odot H \) is an \((m \cdot d^2, \beta \cdot \max(\omega, \beta))\)-graph.
we define the operation. First, an example:

\[ G = \begin{array}{c}
\begin{array}{ccc}
& & \\
& \ast & \\
\ast & & \\
\end{array}
\end{array} \quad H = \begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\end{array} \quad G \odot H = \begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\end{array} \]

First, edges:

more generally, (1) replace each \( G \) vertex by a copy of \( H \)
(2) connect \( G \)-edges to an available slot in the cloud
(3) new edges in \( G \odot H \) are \( H \)-edges

In matrix form:

We define \( \hat{A}_H \) a matrix for blue steps
also \( \hat{A}_G \) a matrix for purple steps

Our final adj matrix for \( G \odot H \) will be \( \hat{A}_H \hat{A}_G \hat{A}_H \)
\[ \widehat{A}_H = \mathbb{I}_{V_6} \otimes A_H \]

\[ \widehat{A}_G ((u_i), (v_j)) = 1 \iff U \nabla V \quad e_i(u) = e_j(v) \]

\[ e_i(u) \text{ is an edge in } G \quad e_1(u) \ldots e_d(u) \]

are the missing edges to \( U \)

\[ A_{G \otimes H} := \mathbb{A}_H \mathbb{A}_G \mathbb{A}_H \quad \text{adj matrix of zigzag product} \]

Why is this a good expander?

Suppose \( S \subseteq V_{G \otimes H} \). If \( S \) splits most clouds \( \rightarrow \) inner edges cross.

If \( S \) is full/empty on most clouds \( \rightarrow G \) edges cross.
Proof of theorem:

**Notation:** For a \( d \)-regular graph, we let \( M = \frac{1}{d} A_G \) be the normalized adjacency matrix.

\[
\langle f, g \rangle := \frac{1}{|V|} \sum_{v \in V} f(v)g(v), \quad \|f\|_1 = \langle f, f \rangle,
\]

\[
\lambda = \max_{\|f\|_1 = 1} \left| \frac{\langle Mf, f \rangle}{\langle f, f \rangle} \right| = \frac{\sum_{i=1}^{n} \alpha_i^2 v_i^2}{\sum_{i=1}^{n} v_i^2} \quad \text{(where } f = \sum_{i=1}^{n} \alpha_i v_i, \lambda = \min(\lambda_1, \lambda_n)\text{)}.
\]

\[
\langle Mf, g \rangle = \mathbb{E}(Mf)(v)g(v) = \mathbb{E}[\mathbb{E}(f(u))g(v)] = \mathbb{E} f(u)g(v) \quad \text{(all edges } u \in E) \]

Fix \( f : V_{G \oplus H} \to \mathbb{R} \) s.t. \( \mathbb{E}_u f(u_i) = 0 \) \( \forall \) \( v \in V_h \)

Define \( f^{\|}(u_i) = \mathbb{E}_{u \in V_{G \oplus H}} f(u) \)

Define \( f^{\perp}(u_i) = f(u_i) - f^{\|}(u_i) \)

\[
\langle f^{\perp}, f^{\|} \rangle = \mathbb{E}_{(u_i)} f^{\perp}(u_i) \cdot f^{\|}(u_i) = \mathbb{E}_{(u_i)} \frac{\mathbb{E} f(u_i)}{\mathbb{E} f^{\perp}(u_i)} \cdot f^{\perp}(u_i) = 0
\]

\[
f = f^{\perp} + f^{\|^}\]

\[
|\langle Mf, f \rangle| = |\langle Mf^{\|}, f^{\perp} \rangle + \langle Mf^{\perp}, f^{\perp} \rangle + \langle Mf^{\|}, f^{\perp} \rangle + \langle Mf^{\perp}, f^{\perp} \rangle|
\]

\[
\leq |\langle Mf^{\|}, f^{\perp} \rangle| + |\langle Mf^{\perp}, f^{\perp} \rangle| + 2 |\langle Mf^{\perp}, f^{\perp} \rangle|
\]
\[ |< \hat{M} f^u, f^u>| = |< A^h f^u, A^h f^u>| \]
\[ = |< A^h f^u, f^u>| = |< M_e f, f>| \]
\[ \leq \lambda_e \cdot \| f^u \|^2 \leq \lambda_e = \alpha \]

\[ \| f^u \|^2 \]

\[ |< M f^\perp, f^\perp>| = |< \hat{M}_e \hat{M}_h f^\perp, \hat{M}_h f^\perp>| \]
\[ \leq \| \hat{M}_e \hat{M}_h f^\perp \| \cdot \| \hat{M}_h f^\perp \| \leq \| \hat{M}_h f^\perp \|^2 \]
\[ \leq \lambda_h \cdot \| f^\perp \|^2 \]
\[ = \beta^2 \cdot \| f^\perp \|^2 \]

Recall \[ \hat{M}_h = M_h \circ I_{V_0} \]
\[ \| \hat{M}_h f^\perp \| \leq \lambda_h \cdot \| f^\perp \| \]
\[ 2 \left| \langle Mf, f' \rangle \right| \leq 2 \| Mf \| \| f' \| \leq 2 \beta \| f' \| \| Mf \| \leq 2 \beta \| f' \| \| f'' \| \leq \frac{s \| f'' \|}{\lambda} \| f' \| \leq \frac{s \| f'' \|}{\lambda} \| f' \|. \]

\[ 0 \| f \|^2 = \| f'' \|^2 + \| f' \|^2 - \langle f', f'' \rangle = 0 \]

\[ \left[ AMGM: \quad 2xy \leq \sqrt{x^2 + y^2} \right] \]

\[ 1 < Mf, f > \leq \alpha \| f' \|^2 + 2 \beta \| f' \| \| f'' \| + \beta^2 \| f'' \|^2 \]

\[ = \beta \| f'' \|^2 + \alpha \left( \| f'' \|^2 - \| f' \|^2 \right) + \beta^2 \| f'' \|^2 \]

\[ \leq \beta \| f'' \|^2 + \max(\alpha, \beta^2) \| f'' \|^2 \]

\hfill \square
Constructing an infinite sequence of \(d\)-regular expander graphs

**Starting point:** take \( H \) to be \((d^4, d^2, \frac{1}{4})\)-graph

- Take \( G_1 = H^2 \) \((d^4, d^2, \frac{1}{16})\)-graph
- \( G_2 = \left( G_1 \right)^2 \otimes H \) \((d^8, d^4, \frac{1}{32})\)-graph
- \( G_{n+1} = \left( G_n \right)^2 \otimes H \)

**Claim:** \( G_n \) is a graph on \(d^{4n}\) vertices, degree \(d^2\),
\[\forall n \geq 1 \quad \max(\lambda_1, \lambda_\ell) \leq \frac{1}{2}\]

\( G_n \) is an \((d^{4n}, d^2, \frac{1}{2})\)-graph

Altogether, the algorithm enumerates graphs to find good \( H \), then proceeds inductively.
Let $G$ be $(n,d,\lambda)$ graph $\lambda \leq 0.1$. Suppose we remove edges from $G$ so that $G' = (V, E')$, $E' \subseteq E$ is $\frac{\lambda}{2}$ regular. Is $G'$ still an expander?