Lesson 9: Applications of ergodic theory

In this section we describe some applications of ergodic theory to number theory.

1 Material

1.1 The pointwise ergodic theorem

We repeat here the pointwise ergodic theorem.

Theorem 19. *Let $T : [0,1) \to [0,1)$ have an absolutely continuous invariant measure ν. Suppose that ν is ergodic for T. Then for every piecewise continuous $f : [0,1) \to [0,\infty)$ there is a set $N_f \subset [0,1)$ of zero measure such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f \, \mathsf{d}\nu$$

for all $x \in [0,1) \setminus N_f$.

If one applies the above version of the pointwise ergodic theorem to the function $f = 1_A$ for $A \subset [0,1)$ an interval then one obtains the pointwise ergodic theorem stated in Lesson 5. Let's state the pointwise ergodic theorem at the level of generality one would usually see it.

Theorem 20. *Fix a probability space $(X, \mathscr{B}, \mu)$ and a measurable map $T : (X,\mathscr{B}) \to (X,\mathscr{B})$ suc that $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathscr{B}$. For every measurable $f : X \to \mathbb{R}$ satisfying*

$$\int |f| \, \mathsf{d}\mu < \infty$$

there is a set $N_f \in \mathscr{B}$ with $\mu(N_f) = 0$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \tag{1}$$

exists for all $x \in X \setminus N_f$. Moreover, if μ is ergodic for T then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f \, \mathsf{d}\mu$$

for all $x \in [0,1) \setminus N_f$.

In particular, the limit exists whether or not we have ergodicity. When the system is ergodic we can easily say what the limit.

1.2 Applications to continued fractions

In Lesson 7 we saw that the Gauss map is ergodic. We can apply ergodicity of the Gauss map to immediately deduce the frequency with which any $b \in \mathbb{N}$ occurs as a digit in the continued fraction expansion of almost every $x \in [0,1)$. Indeed, fixing

$$I = \left[\frac{1}{b+1}, \frac{1}{b}\right)$$

ergodicity gives us a set $N_I \subset [0,1)$ of zero measure such that

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N : a_n(x) = b\}|}{N} = \frac{1}{\log 2} \int\limits_I \frac{1}{x+1} \, \mathsf{d}x$$

holds. In particular, for almost every $x \in [0,1)$ the frequency with which the digit b occurs in the continued fraction expansion of x is

$$\int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{1}{(1+x)\log 2}\,\mathrm{d}x = \frac{1}{\log 2}\log\left(\frac{(b+1)^2}{b(b+2)}\right)$$

so that in particular 1 occurs about 41.5037% of the time whereas 8 occurs only about 1.7922% of the time. It is also possible to prove more delicate results about continued fraction digits. Each of the following results

- $\lim\limits_{n\to\infty} \frac{1}{n}\log q_n(x) = \frac{\pi^2}{12\log 2}$
- $\lim\limits_{n\to\infty} \frac{1}{n}\log \lambda(\Delta_n(x)) = -\frac{\pi^2}{6\log 2}$
- $\lim\limits_{n\to\infty} \frac{1}{n}\log \left| x - \frac{p_n(x)}{q_n(x)} \right| = \frac{-\pi^2}{6\log 2}$

is true for almost every $x \in [0,1)$ and we will see some of the details in the problem set.

A further interesting result about the digits of continued fractions is related to the geometric mean of the continued fraction digits. The **geometric mean** of positive numbers $y_1,\dots,y_n$ is

$$\sqrt[n]{y_1\cdots y_n} = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}$$

and we always have

$$\frac{y_1+\cdots+y_n}{n} \geq \sqrt[n]{y_1\cdots y_n}$$

which is the arithmetic-geometric mean inequality.

Theorem 21. *One has*

$$\lim_{n\to\infty} \sqrt[n]{a_1\cdots a_n} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+1)}\right)^{\frac{\log k}{\log 2}} \approx 2.6854$$

for almost all $x \in [0,1)$.

We will see in the exercises that the arithmetic average

$$\lim_{N\to\infty} \frac{a_1(x)+\cdots+a_N(x)}{N} = \infty$$

for almost all $x \in [0,1)$. Despite the fact that small digits appear far more often in the continued fraction expansion of a typical real number $x \in [0,1)$ large digits appear often enough to make the average of the first N digits diverge.

1.3 Mixing of the Gauss map

We say in the Lesson 5 problem set that the ternary map $T(x) = 3x \bmod 1$ is mixing. That is to say: for any measurable sets $A, B \subset [0,1)$ we have

$$\lim_{n\to\infty} \nu(A \cap (T^n)^{-1}B) = \nu(A)\nu(B)$$

and indeed it suffices to check the above limit in the case that A, B are fundamental intervals.

Is the Gauss map mixing? It is, but it is not as easy to prove as mixing for the ternary map. We will see in the exercises that

$$\nu(\Delta_2(a_1,a_2)) \neq \nu(\Delta_1(a))\nu(\Delta_1(a_2))$$

where was we certainly have

$$\lambda(I_{ij}) = \lambda(I_i)\lambda(I_j)$$

where λ is the invariant measure for the ternary map.

There are several approaches to mixing of the Gauss map, all of which could serve as the "next step" in learning about ergodic theory.

1.1 Analyzing the transfer operator

$$(Gf)(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right)$$

on the right Banach space of functions.

1.2 Proving that the Gauss map is exact, and that exact transformations are mixing. This leads to studying entropy of dynamical systems.

1.3 Studying the invertible extension of the Gauss map.

1.4 Other applications to number theory

We mention here two other applications of ergodic theory to number theory more broadly. These are two of the hallmark contributions of ergodic theory in the past fifty years.

Furstenberg's proof of Szemerédi's theorem A set $E \subset \mathbb{N}$ has positive density if

$$\liminf_{N\to\infty} \frac{|\{1 \le n \le N : n \in E\}|}{N} > 0$$

and one can think of sets of positive density as taking up a positive proportion of the natural numbers.

Theorem 22 (Szemerédi)**.** *If $E \subset \mathbb{N}$ has positive density then for every $k \in \mathbb{N}$ there are $a, d \in \mathbb{N}$ such that $\{a, a+d, \dots, a+kd\} \subset E$.*

Furstenberg gave an alternative proof of Szemerédi's theorem using ergodic theory. Frstenberg proved that the following theorem, and proved that the theorem implies Szemerédi's theorem.

Theorem 23. *For every measure-preserving transformation T of a probability space $(X, \mathscr{B}, \nu)$ and every $B \in \mathscr{B}$ and every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$\nu(B \cap (T^n)^{-1}B \cap \cdots \cap (T^{kn})^{-1}B) > 0 \qquad (2)$$

holds.

In particular, for every $T : [0,1) \to [0,1)$ with an absolutely continuous invariant measure ν and every $B \subset [0,1)$ wih $\nu(B) > 0$ the inequality (2) holds.

The new perspective on density combinatorics afforded by Furstenberg's proof led to many new results, including playing a role in the proof of the Green-Tao theorem that the set of prime numbers contain arithmetic progressions of all lengths.

Margulis's resolution of the Oppenheim conjecture A quadratic form is any map $Q : \mathbb{R}^n \to \mathbb{R}$ of the form $Q(x) = x^{\mathsf{T}}Ax$ for some $n \times n$ symmetric matrix A. For example, if

$$A = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

then

$$Q(x) = x_1(2x_1 + \sqrt{3}x_2) + x_2(\sqrt{3}x_1 + 2x_2) = 2x_1^2 + 2\sqrt{3}x_1x_2 + 2x_2^2$$

The Oppenheim conjecture is about whether the image of $\mathbb{Z}^n$ under Q is dense in $\mathbb{R}$. The **Oppenheim conjecture** is that the following three requirements together imply $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

- A is not a multiple of a matrix with rational entries.
- Q is not definite i.e. there are x, y with $Q(x) > 0$ and $Q(y) < 0$.
- $n \geq 3$.

After some thought, the first two requirements are not so surprising. If A is a multiple of a rational form then $Q(\mathbb{Z}^n)$ cannot be dense in $\mathbb{R}$.

Conjecture 24 (Oppenheim)**.** *If a quadratic form Q in at least three variables is not a multiple of a rational form and is not definite then $Q(\mathbb{R}^n)$ is dense in $\mathbb{R}$.*

This conjecture was resolved by Margulis using homogeneous dynamics; specifically by understanding the ergodic theory of how the group $\mathsf{SO}(2,1)$ acts on the coset space $\mathsf{SL}(3,\mathbb{R})/\mathsf{SL}(3,\mathbb{Z})$.

2 Problems

2.1 Digits of powers

2.1 Let f_n be the left-most digit in the decimal expression of 2^n. For example $f_5 = 3$.

(a) Verify that $f_n = k$ if and only if $\log_{10}(k) \leq \{n \log_{10}(2)\} < \log_{10}(k+1)$.

(b) Verify that $\log_{10}(2)$ is irrational.

(c) Use ergodicity of the irrational rotation $T(x) = x + \log_{10}(2)$ to compute the asymptotic frequency of $n \in \mathbb{N}$ for which f_n is 5.

2.2 Use the ideas in the previous problem to prove that every decimal expression appears as the left-most portion of the decimal representation of some power of two. (Hint: For irrational rotations the conclusion of the pointwise ergodic theorem holds for *all* points in $[0,1)$. Why? Cf. Lesson 5 Problem 2.12.)

2.2 Continued fraction applications

2.3 Take for granted that the pointwise ergodic theorem is true for piecewise continuous functions f. That is, if T is ergodic with respect to an absolutely continuous invariant measure ν and $f : [0,1) \to [0,\infty)$ is piecewise constant then there is $N_f \subset [0,1)$ of zero measure such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f(x)\phi(x)\,\mathsf{d}x$$

holds for all $x \in [0,1) \setminus N_f$.

(a) For T the Gauss map which choice of f makes

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

the average of the digits $a_2, \ldots, a_{N+1}$ in the continued fraction expansion of x?

(b) What does the pointwise ergodic theorem tell us about the average of the continued fraction digits of almost all $x \in [0,1)$?

(c) Which choice of f will tell us about the geometric mean $\sqrt[n]{a_2 \cdots a_{n+1}}$ of the continued fraction digits $a_2, \ldots, a_{n+1}$ of almost all $x \in [0,1)$? What is the result?

2.4 In this problem we will look into the more delicate question of the limit

$$\lim_{n\to\infty} \frac{1}{n} \log q_n$$

for almost all $x \in [0,1)$ where $q_n(x)$ are the denominators of the continued fraction approximants of x. The evaluation of this limit goes back to Lévy in 1929.

(a) Use $p_n(x) = q_{n-1}(Tx)$ from the Lesson 2 exercises to write

$$-\log q_n(x) = \log \frac{p_n(x)}{q_n(x)} + \log \frac{p_{n-1}(T(x))}{q_{n-1}(T(x))} + \cdots + \log \frac{p_1(T^{n-1}(x))}{q_1(T^{n-1}(x))}$$

(b) Why is replacing $-\frac{1}{n}\log q_n(x)$ with

$$\frac{1}{n}\left(\log x + \log(Tx) + \cdots + \log(T^{n-1}x)\right)$$

not entirely unreasonable? (The estimates here are a little involved, with the details below.)

(c) Apply the pointwise ergodic theorem to $f(x) = \log x$ to deduce the limit in $\frac{\pi^2}{12\log 2}$.

In the remaining steps we will estimate the difference

$$R(x,n) = -\log(x) - \log(Tx) - \cdots - \log(T^{n-1}x) - \log q_n(x)$$

and show that it converges to zero after dividing by n.

(d) Let $F_1, F_2, F_3, F_4, F_5, F_6, \dots$ be the Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$. Prove that $q_n(y) \geq F_n$ for all $n \in \mathbb{N}$ and all $y \in [0,1)$.

(e) Use the mean value theorem on an appropriate interval to deduce that

$$0 < \log x - \log \frac{p_n(x)}{q_n(x)} < \frac{1}{q_n(x)}$$

when n is even. Find an analogous statement for n odd.

(f) Conclude that the remainder $R(x,n)$ is bounded by

$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n}$$

for all $n \in \mathbb{N}$ and all $x \in [0,1)$.

(g) Prove that the above series converges using

$$F_n = \frac{\phi^n + (-1)^{n+1}\overline{\phi}^n}{\sqrt{5}}$$

and conclude the proof.

2.3 Szemerédi's theorem

2.5 Let $T : [0,1) \to [0,1)$ be the ternary map $T(x) = 3x \bmod 1$ and let ν be its absolutely continuous invariant measure. Fix $I = [\frac{i}{3^k}, \frac{i+1}{3^k})$ for some $k \in \mathbb{N}$ and some $0 \leq i < 3^k$. Prove that

$$\lim_{n\to\infty} \nu(I \cap (T^n)^{-1} I \cap \cdots \cap (T^{dn})^{-1} I) = \nu(I)^{d+1}$$

for all $d \in \mathbb{N}$.

2.6 Fix $\alpha \in \mathbb{R}$ irrational. Let $T : [0,1) \to [0,1)$ be the irrational rotation $T(x) = x + \alpha \bmod 1$. Fix an interval $I \subset [0,1)$. Prove that

$$\limsup_{n\to\infty} \nu(I \cap (T^n)^{-1} I \cap \cdots \cap (T^{dn})^{-1} I) = \nu(I)$$

for all $d \in \mathbb{N}$.

2.7 Fix $\alpha \in \mathbb{R}$ and an interval $A \subset [0,1)$. Let $T : [0,1) \to [0,1)$ be the map $T(x) = x + \alpha \bmod 1$.

(a) Verify that the set
$$R = \{n \in \mathbb{N} : T^n(0) \in A\}$$
has positive density.

(b) Prove that R contains arithmetic progressions of all lengths.

2.8 Verify that the set of primes does not have positive density by proving the following upper bound of Chebyshev. Define ϑ by
$$\vartheta(x) = \sum_{p \le x} \log p$$
where the sum is over all primes not larger than $x \in \mathbb{N}$.

(a) Prove for all $m \in \mathbb{N}$ that
$$\prod_{m \le p \le 2m} p \le \binom{2m}{m} \le 2^{2m}$$
where the product is restricted to p prime.

(b) Every $x \in \mathbb{N}$ lies in some interval $(2^i, 2^{i+1}]$. Combine with an estimate for $\vartheta(2m) - \vartheta(m)$ using the previous step to show that $\vartheta(x) \le 2x$ for all $x \in \mathbb{N}$.

(c) Write $\pi(x)$ for the number of primes less than or equal to $x \in \mathbb{R}$. Prove that
$$\pi(x) \le \pi(\sqrt{x}) + \frac{1}{\log \sqrt{x}} \sum_{\sqrt{x} < p \le x} \log p$$
for all x.

(d) Trivially estimate $\pi(\sqrt{x})$ and combine with (b) to deduce that $\pi(x) \log x \le 4x$.

(e) Use this to prove the primes do not have positive density.

2.4 The Oppenheim conjecture

2.9 A real number a is **badly approximable** if there is a constant $c > 0$ such that
$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}$$
for all $\frac{p}{q} \in \mathbb{Q}$.

(a) Prove (or recall from Lesson 8) that the golden ration $\phi = (1 + \sqrt{5})/2$ is badly approximable.

(b) Prove for the quadratic form $Q(x, y) = x^2 - \phi^2 y^2$ that $Q(\mathbb{Z}^2)$ is not dense in $\mathbb{R}$.

2.10 Fix the quadratic form $P(x, y, z) = x^2 + y^2 - z^2$.

(a) Describe the set $\mathsf{SO}(P)$ of matrices $A \in \mathsf{SL}(3, \mathbb{R})$ with the property that $P(Av) = P(v)$ for all $v \in \mathbb{R}^3$.

Fix a quadratic form Q in three variables that satisfies the hypothesis of the Oppenheim conjecture.

(b) Prove that there is $\lambda \in \mathbb{R}$ and $g \in \mathsf{SL}(3, \mathbb{R})$ such that $Q = \lambda(P \circ g)$.

(c) Prove that $\mathsf{SO}(Q) = g\mathsf{SO}(P)g^{-1}$.

(d) Prove that if $\mathsf{SO}(Q)\mathbb{Z}^3$ is dense in $\mathbb{R}^3$ then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

(e) Describe how $\mathsf{SL}(3, \mathbb{R})/\mathsf{SL}(3, \mathbb{Z})$ is the space of **unimodular lattices** i.e. lattices in $\mathbb{R}^3$ of the form $g\mathbb{Z}^3$ for some $g \in \mathsf{SL}(3, \mathbb{R})$.

(f) Prove that if the $\mathsf{SO}(P)$ orbit of the coset $g\mathsf{SL}(3, \mathbb{Z})$ is dense in $\mathsf{SL}(3, \mathbb{R})/\mathsf{SL}(3, \mathbb{Z})$ then $P(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

Lesson 10: Continued fractions and Diophantine approximation

1 Material

Diophantine Approximation is a branch of number theory concerned with the approximation of real numbers with rational numbers. In Lesson 8, we discussed the existence of rational solutions p/q to the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}, \tag{1}$$

which measures how well p/q approximates x as a function of the height q of the rational number. Dirichlet's theorem told us that for all irrational numbers x, there are infinitely many rational numbers p/q satisfying (1), but the theorem didn't characterize such rational numbers or give us an efficient way to find them.

Thus, we were left with the questions:

(I) Which rational numbers satisfy inequalities such as the one in (1)?

(II) How do we find such rational approximations?

The theory of continued fractions provides a remarkably satisfactory answer to both of these questions, as we will see in this lesson.

From now on, whenever we write p/q, it is tacitly assumed that $q > 0$ and p and q are coprime so that p/q is in lowest terms. The number $x \in \mathbb{R}$ will always be irrational and will have continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$. The n^{th} convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ will always be denoted by p_n/q_n.

1.1 Continued fraction convergents as good rational approximations

The first theorem tells us how well continued fraction convergents approximate real numbers.

Theorem 25. *For all $n \geq 1$,*

$$\frac{1}{(a_{n+1}+2)q_n^2} < \frac{1}{q_n(q_n+q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1}q_n^2} \tag{2}$$

Proof. The first and fourth inequalities follow easily from the fact that $q_{n+1} = a_{n+1}q_n + q_{n-1}$ and the fact that the sequence $n \mapsto q_n$ is positive and increasing.

For the second and third inequalities, we will use the fact that $|p_{n+1}q_n - p_n q_{n+1}| = 1$ and the fact that the sequence

$$\frac{p_n}{q_n}, \quad \frac{p_{n+2}}{q_{n+2}} = \frac{p_n + a_{n+2}p_{n+1}}{q_n + a_{n+2}q_{n+1}}, \quad x, \quad \frac{p_{n+1}}{q_{n+1}}$$

is either increasing or decreasing.

The third inequality follows immediately on noting that

$$\left| x - \frac{p_n}{q_n} \right| < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}}.$$

To see the second inequality, we will show first that $(p_n + p_{n+1})/(q_n + q_{n+1})$ falls between p_n/q_n and x. The convergent p_{n+2}/q_{n+2} is gotten by taking the mediant[5] of p_n/q_n with p_{n+1}/q_{n+1} exactly a_{n+2} many times, each successive mediant lying between the previous one and p_{n+1}/q_{n+1}. Since $(p_n+p_{n+1})/(q_n+q_{n+1})$ is the first of these mediants, it must fall between p_n/q_n and x. The second inequality now follows easily by noting that

$$\left| x - \frac{p_n}{q_n} \right| > \left| \frac{p_n + p_{n+1}}{q_n + q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n(q_n + q_{n+1})}.$$

This concludes the proof of the theorem. □

[5] The **mediant** of a/b and c/d, where $b, d > 0$, is $(a+c)/(b+d)$; it is an exercise to show that it lies between a/b and c/d.

A basic conclusion from Theorem 25 is that *continued fraction convergents of any real number x provide good rational approximations to x.* Indeed, since $a_{n+1} \geq 1$, we see from (2) that every convergent satisfies the inequality from Dirichlet's theorem in (1). Thus, finding good rational approximations to an irrational number can be done using the Euclidean Algorithm via continued fractions!

Another conclusion from Theorem 25 is that *the partial quotients $a_1, a_2, \ldots,$ are intimately related to the degree to which the n^{th} convergent approximates x.* Indeed, it follows from (2) that a_{n+1} is determined to within one integer by the quantity $1/(q_n\|q_nx\|)$. Thus, the partial quotient a_{n+1} is very large if and only if the difference $|x - p_n/q_n|$ is very small. This leads to a very natural description of the set of badly approximable numbers that is explored in the exercises.

Finally, the following theorem is a strengthening of Dirichlet's theorem that follows quickly as a corollary of Theorem 25.

Theorem 26. *For all irrational numbers x and all $n \in \mathbb{N}$, at least one of p_n/q_n or p_{n+1}/q_{n+1} satisfies the inequality*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

In particular, there are infinitely many rational solutions to this inequality.

Proof. The real number x falls between p_n/q_n and p_{n+1}/q_{n+1}. If the conclusion of the theorem did not hold, then

$$\frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} \leq \left|x - \frac{p_n}{q_n}\right| + \left|x - \frac{p_{n+1}}{q_{n+1}}\right| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_nq_{n+1}}.$$

This simplifes to the inequality $(q_n - q_{n+1})^2 \leq 0$, which fails since $q_n < q_{n+1}$. This contradiction means in fact the conclusion must hold. $\square$

Going further, it is a famous theorem of Borel that at least one of every three consecutive continued fraction convergents satisfies

$$\left|x - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Thus, continued fraction convergents also give us an efficient way to find solutions to Hurwitz' strengthening of Dirichlet's approximation theorem that we saw in Lesson 8.

1.2 Continued fraction convergents as the best rational approximations

Now that we have seen that continued fraction convergents provide *good* rational approximates, we take up the matter of whether they are the *best* rational approximates.

Call p/q a **best approximate to** x if it minimizes the distance to x amongst all rational numbers with height at most q. We will see in this section that

(I) all continued fraction convergents of x are best approximates to x;

(II) not all best approximates to x are continued fraction convergents of x;[6]

(III) if p/q is a very good approximate to x in the sense that $|x - p/q| < 1/(2q^2)$, then p/q is a continued fraction convergent of x.

The statement in (II) indicates some subtlety in the matter, and it is this subtlety that causes the statement of the next theorem to look, at first sight, unnatural.

Theorem 27 (Law of Best Approximates)**.** *Of all quantities $|qx - p|$ with $1 \leq q < q_{n+1}$ and $p \in \mathbb{Z}$, the quantity $|q_nx - p_n|$ is nearest to zero.*

You will prove Theorem 27 in the exercises. Before we derive the promised corollaries from it, let us see how it can be used to characterize the continued fraction convergents of an irrational number x. Call an increasing sequence of positive integers $(n_i)_{i\in\mathbb{N}}$ a **sequence of records for** x if

[6] It is true, however, that every best approximate to x is derived from a convergent in some sense; see [Khi2] for more details.

- $n_1 = 1$;
- $\|n_1 x\| > \|n_2 x\| > \|n_3 x\| > \cdots$; and
- for all $n_i < n < n_{i+1}$, $\|nx\| > \|n_i x\|$.

The Law of Best Approximates gives us the following remarkable fact: *the sequence of denominators $(q_n)_{n\in\mathbb{N}}$ of convergents of x is a sequence of records for x.* The terminology comes from keeping track of the first n's that break the previous record by being closest to zero on the circle modulo 1. This characterizes continued fraction denominators, and hence numerators and partial quotients, as "record setting times" of the orbit of zero under a circle rotation.

Theorem 28. *Of all fractions p/q with $q \leq q_n$, the convergent p_n/q_n is nearest to x.*

This theorem follows quickly from Theorem 27, as you will show in the exercises. It says that every continued fraction convergent of x is a best approximate to x. But it is instructive to see what it does not say: it does not say that every best approximate to x is a continued fraction convergent. You will find examples in the exercises.

Theorem 27 also allows us to show that the only very good rational approximations to x are its continued fraction convergents. The following theorem makes this precise.

Theorem 29 (Legendre's theorem)**.** *If*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}, \tag{3}$$

then p/q is a continued fraction convergent of x.

Again, you will work through a proof in the exercises. And again, it is instructive to understand what Legendre's theorem does not say: it does not say that all continued fraction convergents satisfy (3). In fact, as an exercise, you are tasked with finding some that do not.

1.3 The typical rate of approximation

The inequalities in Theorem 25 combine with the Ergodic Theorem to produce a marvelous result concerning the rate of convergence of continued fraction convergents to the typical real number.

Theorem 30 (Lévy-Khinchin theorem)**.** *For Lebesgue almost every $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| = -\frac{\pi^2}{6\log 2}$$

Proof. We have from Theorem 25 that

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{4}$$

Thus, we can estimate the distance between x and the n^{th} continued fraction convergent p_n/q_n as a function of the growth rate of the convergents' denominators.

It is a fact provable using the Ergodic Theorem [EW, Corollary 3.8] that for Lebesgue almost every $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12\log 2}.$$

Thus, for a typical real number, the sequence of continued fraction convergent denominators grows exponentially with base $e^{\pi^2/12\log 2} \approx 3.276$. The conclusion of the Lévy-Khinchin theorem follows by a short calculation from combining the inequalities in (4) with this fact. $\square$

The Lévy-Khinchin theorem says that for a typical real number x, there exists a sequence $(\epsilon_n)_{n=1}^{\infty}$ tending to zero such that for all $n \in \mathbb{N}$,

$$\left| x - \frac{p_n}{q_n} \right| = e^{(-\pi^2/6\log 2 + \epsilon_n)n}.$$

Thus, typically, for large n, the n^{th} partial quotient of x is at a distance of about $e^{-2.373n}$ from x. The number $e^{-\pi^2/6\log 2}$ is just less than $1/10$, so typically, for large n, the n^{th} approximates x to within at least n many decimal places.

2 Problems

2.1 Best approximations to π

*2.1 Compute the first few continued fraction convergents to π. How do these compare to the infamous $22/7$?

2.2 Find a few best approximations to π that are not a continued fraction convergents of π.

2.3 Find some continued fraction convergents to π that do not satisfy the inequality in (3).

2.4 ([Loy]) Let $\varphi : \mathbb{N} \to (0,1)$ be a function. Show that there exists an irrational number ξ and infinitely many rationals p/q satisfying

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{\varphi(q)}.$$

Combine this theorem with Liouville's theorem to easily produce transcendental numbers.

2.2 Proof of Theorem 27, the Law of Best Approximates

This sequence of exercises comprises a complete proof of Theorem 27.

2.5 Argue that the following two statements suffice to prove the theorem.

(I) The sequence $n \mapsto |q_n x - p_n|$ is decreasing.

(II) If $p \in \mathbb{Z}$ and $q \geq 1$ minimize the expression $|q'x - p'|$ amongst all $p' \in \mathbb{Z}$ and $1 \leq q' \leq q$, then p/q is a continued fraction convergent of x.

2.6 Use the inequalities in Theorem 25 to show that the sequence $n \mapsto |q_n x - p_n|$ is decreasing. This establishes (I).

Establishing (II) is a bit more challenging. Suppose that $p \in \mathbb{Z}$ and $q \geq 1$ minimize the expression $|q'x - p'|$ amongst all $p' \in \mathbb{Z}$ and $1 \leq q' \leq q$. Suppose also that p/q is not a convergent of x. We aim to reach a contradiction.

2.7 Show that if $p/q < a_0$, then $|x - a_0| < |qx - p|$, contradicting the assumption on p and q.

2.8 Show that if $p/q > p_1/q_1$, then $|x - a_0| < |qx - p|$, contradicting the assumption on p and q. (Hint: Show that $|x - p/q| > |p_1/q_1 - p/q| \geq 1/(qq_1)$ and that $|x - a_0| \leq 1/a_1$, then combine these facts.)

2.9 Conclude from the previous two exercises and the alternating property of the convergents that p/q must lie between two convergents p_{m-1}/q_{m-1} and p_{m+1}/q_{m+1} for some $m \geq 0$.

2.10 Show that

$$\frac{1}{qq_{m-1}} \leq \left| \frac{p}{q} - \frac{p_{m-1}}{q_{m-1}} \right| \leq \left| \frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}} \right| = \frac{1}{q_{m-1}q_m},$$

and conclude that $q > q_m$.

2.11 Show that

$$\frac{1}{qq_{m+1}} \leq \left|\frac{p_{m+1}}{q_{m+1}} - \frac{p}{q}\right| \leq \left|x - \frac{p}{q}\right|,$$

and combine this with Theorem 25 to conclude that

$$|q_m x - p_m| \leq \frac{1}{q_{m+1}} \leq |qx - p|.$$

2.12 Combine **2.10** and **2.11** to reach a contradiction with the assumptions on p and q and conclude the proof of Theorem 27.

2.3 Proof of Theorem 28

2.13 Suppose that p/q is the rational number with $q \leq q_n$ that is nearest to x. Show that if $q = q_n$, then $p/q = p_n/q_n$.

2.14 Show that if $q < q_n$, then $|x - p/q| < |x - p_n/q_n|$. Conclude that $|qx - p| < |q_n x - p_n|$.

2.15 Combine the previous exercise with Theorem 27 to reach a contradiction and conclude the proof of Theorem 28.

2.4 Proof of Theorem 29, Legendre's theorem

2.16 Argue using Theorem 27 that Legendre's theorem follows from the statement: if $|x - p/q| < 1/(2q^2)$ and the rational r/s is such that $|sx - r| < |qx - p|$, then $s > q$.

2.17 Suppose that $|x - p/q| < 1/(2q^2)$ and that the rational r/s is such that $|sx - r| < |qx - p|$. Show that $|x - r/s| < 1/(2qs)$.

2.18 Use the triangle inequality and the fact that $r/s \neq p/q$ to show that

$$\frac{1}{qs} < \left|\frac{r}{s} - \frac{p}{q}\right| < \frac{1}{2q^2} + \frac{1}{2qs}.$$

2.19 Conclude from the previous problem that $s > q$, concluding the proof of Legendre's theorem.

2.5 Badly approximable numbers

Recall from Lesson 8 that an irrational number x is **badly approximable** if there exists $c > 0$ such that for all p/q,

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

2.20 Use Theorem 25 to show that x is badly approximable if and only if it has "bounded partial quotients," that is, the sequence of its partial quotients is bounded.

2.21 It is a fact that a real number x is a quadratic irrational if and only if the sequence of its partial quotients is eventually periodic. Observe that it follows immediately from this fact that all quadratic irrationals are badly approximable.

2.22 Show that the set of badly approximable numbers is uncountable.

2.23 Use the Ergodic Theorem to show that the set of badly approximable numbers in $[0, 1]$ has zero measure with respect to the Gauss measure. Conclude that the set of badly approximable numbers has zero Lebesgue measure.

2.24 Show that the set of pairs (x, y) that do not satisfy the Littlewood conjecture (from Lesson 8) has zero Lebesgue measure in the plane.

References

[BvdPSZ] Jonathan Borwein, Alf van der Poorten, Jeffrey Shallit, and Wadim Zudilin. *Neverending fractions*, volume 23 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2014. An introduction to continued fractions.

[DK] Karma Dajani and Cor Kraaikamp. *Ergodic theory of numbers*, volume 29 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 2002.

[EW] Manfred Einsiedler and Thomas Ward. *Ergodic theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[Khi1] A. Khintchine. Metrische Kettenbruchprobleme. *Compositio Math.*, 1:361–382, 1935.

[Khi2] A. Ya. Khintchine. *Continued fractions*. Translated by Peter Wynn. P. Noordhoff, Ltd., Groningen, 1963.

[Lév] P. Lévy. Sur les lois de probabilité dont dependent les quotients complets et incomplets d'une fraction continue. *Bull. Soc. Math. France*, 57:178–194, 1929.

[Loy] Paul Loya. *Amazing and aesthetic aspects of analysis*. Springer, New York, 2017.

[Ser] Caroline Series. The modular surface and continued fractions. *J. London Math. Soc. (2)*, 31(1):69–80, 1985.