Ergodic Theory with Applications to Continued Fractions

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Daniel Glasscock
University of Massachusetts Lowell

Claire Merriman
The Ohio State University

Donald Robertson
University of Manchester

Clifford Smyth
University of North Carolina Greensboro

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Lesson 1: Digits and dynamics

1 Material

Lesson 1 introduces some of the basic concepts in dynamical systems in the context of the $x \mapsto 3x \pmod 1$ map; introduces some of the basic concepts of the representations of numbers in the context of ternary expansions; and demonstrate the connection between the two.

Throughout these notes, $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of positive integers. A sequence of elements of a set X is a function from $\mathbb{N}$ to X. Such a function is frequently denoted by $x_1, x_2, \ldots$; by $(x_n)_{n\in\mathbb{N}}$; by $n \mapsto x_n$; or by $x \in X^{\mathbb{N}}$.

1.1 Digits

We can understand the set of real numbers abstractly as the completion of the rational numbers with respect to the Euclidean metric. As a practical matter, however, it is useful to be able to represent real numbers in terms of integers or rational numbers. Each such **representation of real numbers** has the potential to teach us something new in number theory.

Every real number $x \in [0,1]$ can be written in **ternary** (**base three**) as

$$x = \sum_{n=1}^{\infty} \frac{d_n}{3^n} = (0.d_1 d_2 d_3 \ldots)_3,$$

where $d_n = d_n(x) \in \{0,1,2\}$ is the $n^{\textbf{th}}$ **digit** of x. The first digit tells us which of the **fundamental intervals**

$$I_0 = [0, 1/3], \quad I_1 = [1/3, 2/3], \quad I_2 = [2/3, 1]$$

contains the number x. If each fundamental interval is divided into thirds, so that, for example, I_2 is divided into

$$I_{20} = [2/3, 7/9], \quad I_{21} = [7/9, 8/9], \quad I_{22} = [8/9, 1],$$

then the first and second digits taken together tell us which of the 9 fundamental intervals of length $1/9$ contain x. This is drawn out in Fig. 1. The fundamental interval I_{10112} has length 3^{-5} and consists of exactly those real numbers whose first five ternary digits are 1, 0, 1, 1, and 2. If we know the first n digits of x in ternary, we can locate x on the number line within a distance of 3^{-n}. If we know all the digits of x in ternary, we can locate x on the number line exactly.[1]

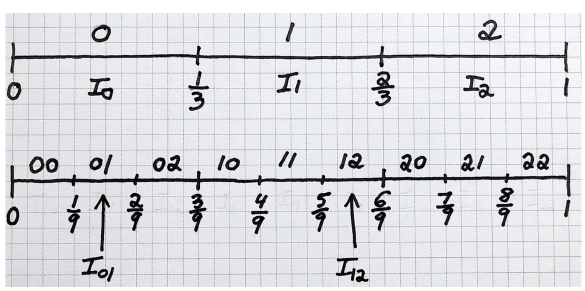


Figure 1: The first digits of a number in ternary locate it in the interval $[0, 1)$.

Many questions arise naturally:

(I) Which real numbers have unique ternary expansions? Which rational numbers have finite, periodic, or eventually periodic ternary expansions?

[1] The number x is the unique member of the intersection of the decreasing sequence of closed intervals. That this intersection is non-empty is guaranteed by the completeness of the real numbers.

(II) If we choose a real number at random, will it have arbitrarily long strings of 1's in its ternary expansion? Does the real number $1/\sqrt{2}$ have this property?

(III) How many real numbers have ternary expansions avoiding the digit 1? How many real numbers have exactly 1/3 of their ternary digits equal to 1? (And what do "how many" and "exactly 1/3" mean?)

(IV) Is there any correlation between the digits of $1/\sqrt{2}$ and $1/\sqrt{3}$ in ternary?

We can answer some of these questions by understanding more clearly the connection between fundamental intervals and ternary digits illustrated in Fig. 1. The ternary digits of $x \in [0,1]$ can be understood as commands to select successively the left hand (digit 0), middle (digit 1), or right hand (digit 2) intervals. The sequence of commands "middle, middle, middle, ..." corresponds to the number 1/2, which is one explanation as to why

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

We could read the ternary expansion $x = (0.0112\ldots)_3$ as "left, middle, middle, right, ...," indicating that x is in the left subinterval $[0, 1/3]$, then x is in the next-smaller middle subinterval $[1/9, 2/9]$, then x is in the next-smaller middle subinterval $[1/9 + 1/27, 1/9 + 2/27]$, and so on. We would know that $x \in I_{0112}$, where I_{0112} is the subinterval of length 3^{-4} that corresponds to the finite sequence of directions "left, middle, middle, right."

It is clear then that two different sequences of commands can lead to the same number if and only if the number lies at the endpoints of one of the fundamental intervals. In this case, for example, the commands "left, right, right, right, ..." and "middle, left, left, left, ..." lead to the same number 1/3. The set of endpoints of the fundamental intervals is the set of rational numbers with denominator a power of three, and those are the only real numbers with two different ternary expansions: one ending in an infinite string of 0's (usually, these expansions are called finite) and one ending in an infinite string of 2's. Let us agree that whenever we need to make a choice between these two, we will automatically choose the finite expansion.

We have answered part of question (I), but answering the other questions in this manner would be tedious at best and impossible at worst.

1.2 Dynamics

Dynamical systems is a subfield of analysis concerned with the behavior of points and sets under iterates of maps. To begin, at the very least, we need a **space** X and a **self-map** $T : X \to X$. This basic framework already allows us to ask about the trajectory of a point $x \in X$ under the map T, its **forward-orbit** is the sequence Tx, T^2x, T^3x, ... Some notation: when it is clear, we frequently omit parenthesis and write Tx instead of $T(x)$ and $T^n x$ instead of $T(T(\cdots T(x)\cdots))$.

Consider the space $X = [0,1)$ and the map $T_3 : X \to X$ defined by $T_3 x = \{3x\}$. Here $\{\cdot\} : \mathbb{R} \to [0,1)$ is the map which takes a real number x and returns its fractional part $\{x\}$. For example, $T_3(4/5) = \{12/5\} = \{2/5\}$.

Many interesting questions arise even at this very early stage:

(I) Which points $x \in X$, if any, are **fixed points** of T_3: $T_3 x = x$? More generally, which are **periodic**: there exists $1 \le n$ such that $T_3^n x = x$? Even more generally, which are **eventually periodic**: there exist $1 \le m < n$ such that $T_3^n x = T_3^m x$? If there are any such points, how many are there? Is there an effective way to find them all? Is there a nice classification of such points?

(II) Which points $x \in X$ are **recurrent**: for any open interval I containing x, there exists $n \in \mathbb{N}$ such that $T_3^n x \in I$? If a point recurs, how often does it recur?

(III) Which points $x \in X$ have a **dense orbit**: the set $\{T_3 x, T_3^2 x, \ldots\}$ is dense in X? Does **almost every** point have a dense orbit: does the set of points without a dense orbit have measure zero (see Lesson 3)?

(IV) How much time does a point $x \in X$ spend in the interval $[1/3, 2/3]$: what proportion of $0 \le n \le N$ are such that $T_3^n x \in [1/3, 2/3]$? Is the sequence $n \mapsto T_3^n\big(1/\sqrt{2}\big)$ **equidistributed**: does the orbit of $1/\sqrt{2}$ spend $(b-a)$ proportion of the time in the interval $[a,b]$ for every $0 \le a < b < 1$?

(V) How close do two points $x, y \in X$ come to one another along their orbits: what is $\inf_{n\in\mathbb{N}} |T_3^n x - T_3^n y|$? Does the point $(x, y) \in X \times X$ have dense orbit under $T_3 \times T_3$: is the set $\{(T_3 x, T_3 y), (T_3^2 x, T_3^2 y) \ldots\}$ dense in $X \times X$?

We can get a better feeling for the map T_3 and answer some of these questions by looking at its graph. Consider Fig. 2. The graph of the function $x \mapsto 3x$ on the interval $[0, 1]$ is drawn on the left. To recover the graph of T_3, we need only to remove the integer part from the range.

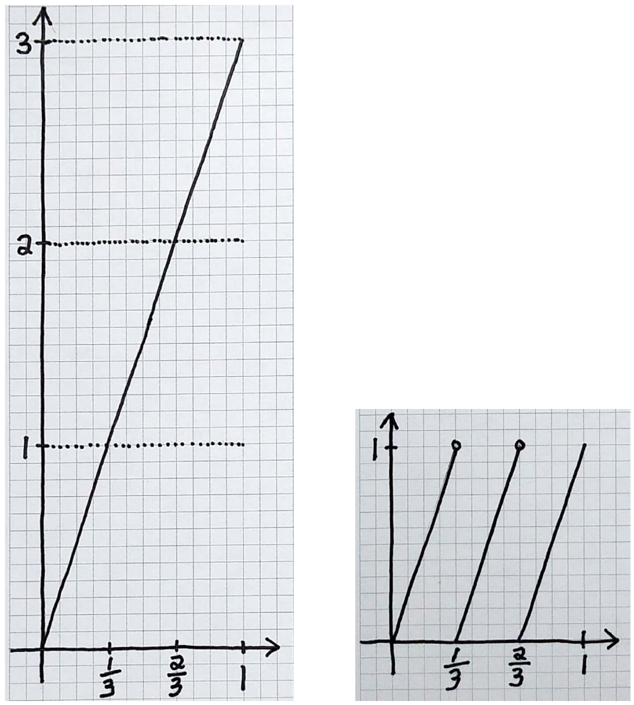


Figure 2: Graphs of the functions $x \mapsto 3x$ and $x \mapsto \{3x\}$.

There is a convenient way to visualize the orbit of a point using the graph of the map T_3. See Fig. 3. We sketch the line $x = y$ atop the graph of T_3 and will visualize the orbit of $x \in [0, 1)$ on this line. Beginning at (x, x), we trace vertically to the graph of T_3 then horizontally back to the $x = y$ line. The new point is $(T_3 x, T_3 x)$. The sequence of points on the line $y = x$ we generate with this procedure is precisely the orbit of x (visualized on the line $x = y$). The orbit of the point $7/12$ is visualized in Fig. 3. What does the orbit of an eventually periodic look like in such a picture? What would a point with a dense orbit look like?

1.3 Connection

The orbit of a real number $x \in [0, 1)$ under the map T_3 and the ternary digits of x are intimately related. Once we understand their relationship, we will see how many of the questions posed above – difficult to answer in their own context – become much more natural to approach.

Writing $x \in [0, 1)$ in ternary then applying the map T_3, we see that

$$T_3 x = T_3\left(\sum_{n=1}^{\infty} \frac{d_n(x)}{3^n}\right) = \left\{\sum_{n=1}^{\infty} \frac{d_n(x)}{3^{n-1}}\right\} = \left\{d_1(x) + \sum_{n=1}^{\infty} \frac{d_{n+1}(x)}{3^n}\right\} = \sum_{n=1}^{\infty} \frac{d_{n+1}(x)}{3^n}.$$

Thus, the first ternary digit of $T_3 x$ is the second ternary digit of x, the second digit of $T_3 x$ is the third digit of x, and so on. To put this another way,

$$T_3\big(0.d_1(x)d_2(x)d_3(x)\cdots\big)_3 = \big(0.d_2(x)d_3(x)d_4(x)\cdots\big)_3. \tag{1}$$

The map T_3 forgets the first ternary digit of x and *shifts* the remaining ternary digits of x "to the left."

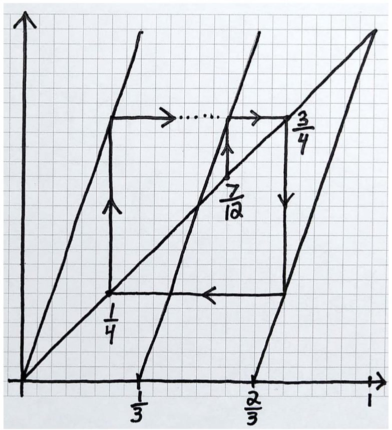


Figure 3: Using the graph of T_3 to visualize the orbit of the point $7/12$ under T_3.

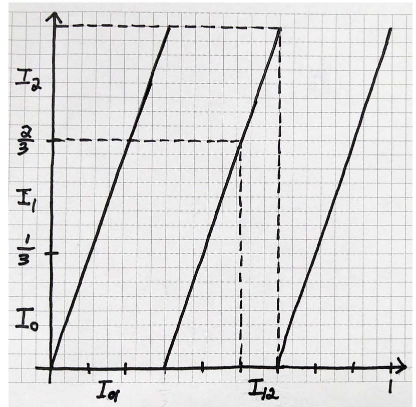


Figure 4: $T_3 I_{12} = I_2$.

There is a nice pictorial way to understand (1). Fig. 4 shows why $T_3 I_{12} = I_2$. This says that if the first two ternary digits of x are 1 and 2, then the first ternary digit of $T_3 x$ is 2. In fact, because the map T_3 restricted to I_1 is linear, the same picture shows that $T_3 I_{1202} = I_{202}$. The map T_3 is 3-to-1, and it demonstrates the same behavior on the intervals I_0 and I_2 as it does on I_1. Thus, $T_3 I_{0112} = I_{112}$ and $T_3 I_{21002} = I_{1002}$, for example. This understanding reinforces (1): the map T_3 forgets the first ternary digit of a fundamental interval and shifts the remaining digits to the left.

Now we are in a position to see how the ternary digits of x help us to understand the orbit of x under T_3. Applying (1) repeatedly, we see that

$$T_3^n \big(0.d_1(x)d_2(x)d_3(x)\cdots\big)_3 = \big(0.d_{n+1}(x)d_{n+2}(x)d_{n+3}(x)\cdots\big)_3.$$

Thus, for example, the n^{th} point in the orbit of x under T_3 is $T_3^n x$, and this point falls into I_2 if and only if $(n+1)^{\text{st}}$ ternary digit of x is equal to 2. Extending this reasoning, we have the following fundamental fact.

Ternary digits and T_3 orbit. $T_3^n x \in I_{i_1 \cdots i_k}$ if and only if $d_{n+1}(x) = i_1, \ldots, d_{n+k}(x) = i_k$.

Thus, the 10^{th} and 11^{th} digits of x are 2 and 0, respectively, if and only if $T_3^9 x \in [2/3, 7/9]$. We will see in the problem set how digits help us answer dynamical questions, and we will see later on in the school how dynamics helps us answer digit questions. To whet your appetite: if we know that the orbit of almost every point spends $1/3$ of its time in the interval $[1/3, 2/3]$, then we can conclude that almost all numbers have $1/3$ of their ternary digits equal to 1!

Understanding the map T_3 as a shift on the ternary digits of the real numbers led to the development of a more abstract topic called **symbolic dynamics**. As the name suggests, symbolic dynamics concerns the behavior of infinite strings of symbols under the shift map. This naive-sounding idea has had a great influence on the field of dynamics. We will explore the beginnings of symbolic dynamics in the problem set.

2 Problems

2.1 Orbits of points under T_3

2.1 Give an example of a point with an eventually periodic T_3-orbit of period 7.

2.2 Show that the set of periodic points of T_3 is dense in $[0, 1]$.

2.3 Give an example of a real number whose T_3-orbit avoids the interval $[0.12, 0.46]$.

2.4 Give an example of a real number with a dense T_3-orbit. Conclude (easily) that the set of points with a dense T_3-orbit is dense.

*__2.5__ Show that the set of points with a dense T_3-orbit is **residual**, an intersection of countably many sets with dense interior.

2.6 Give an example of a real number that is aperiodic under T_3 but that visits the intervals I_0, I_1, and I_2 asymptotically equally often.

2.7 Formulate a necessary and sufficient condition on the ternary digits of an irrational number for it to be recurrent under T_3. Give an example of a real number that is aperiodic under T_3 and does not have a dense T_3-orbit, but is nevertheless recurrent under T_3.

2.8 Show that if I is a fundamental interval, then $T_3 I$ is triple the length of I while $T_3^{-1} I$ is a union of three intervals of total length equal to that of I. (The importance of this will become clear in Lesson 3.)

2.2 Dynamics under the map $T_3 \times T_3$

Given $T : X \to X$, the map $T \times T : X^2 \to X^2$ is defined by $(T \times T)(x, y) = (Tx, Ty)$.

2.9 If x has a dense T_3-orbit in $[0, 1)$, does the point (x, x) have a dense $(T_3 \times T_3)$-orbit in $[0, 1)^2$? Give an example of a pair of real numbers (x, y) whose $(T_3 \times T_3)$-orbit is dense in $[0, 1)^2$.

2.10 Suppose x and y both visit the intervals I_i, $i = 0, 1, 2$, asymptotically equally often under T_3. Is it true that the point (x, y) visits the sets $I_i \times I_j$, $i, j = 0, 1, 2$, asymptotically equally often under $T_3 \times T_3$?

After solving the following problems, interpret their statements as ones about the dynamics of a point $(x, y) \in [0, 1)^2$ in relation to the diagonal $\{(x, x) \mid x \in [0, 1)^2\}$.

2.11 Give an example of a pair of distinct real numbers x and y such that for all $n \in \mathbb{N}$, $T_3^n x \neq T_3^n y$, but such that $\inf_{n \in \mathbb{N}} |T_3^n x - T_3^n y| = 0$. (The points x and y are **proximal**.)

2.12 Show that if x and y are distinct, then there exists $n \in \mathbb{N}$ such that $|T_3^n x - T_3^n y| \geq 1/9$. (The map T_3 is **expansive**.)

2.3 Cantor's middle thirds set

Let $C \subseteq [0, 1]$ be **Cantor's middle thirds set**,

$$
\begin{aligned}
C &= \big(I_0 \cup I_2\big) \cap \big(I_{00} \cup I_{02} \cup I_{20} \cup I_{22}\big) \cap \big(I_{000} \cup I_{002} \cup I_{020} \cup I_{022} \cup I_{200} \cup I_{202} \cup I_{220} \cup I_{222}\big) \cap \cdots \\
&= \bigcap_{\ell=1}^{\infty} \bigcup_{\omega \in \{0,2\}^\ell} I_\omega,
\end{aligned}
$$

where $\{0, 2\}^\ell$ is the set of words of length ℓ in the digits 0 and 2.

2.13 Show that the set C is the set of real numbers in $[0, 1]$ that can be expressed in ternary with only the digits 0 and 2.

2.14 Show that the set C is compact and uncountable.

2.15 Show that the set C is T_3**-invariant**: $T_3 C = C$. (Thus, we can ask all the same dynamical questions about $T_3 : C \to C$, but now relative to C instead of $[0, 1)$.)

2.16 Show that the T_3-orbit of no point in C is dense in $[0, 1]$. Exhibit uncountably many points in C whose T_3-orbit is dense in C.

*__2.17__ Give an example of a closed, T_3-invariant subset of the set C that is not equal to $\emptyset$ or C.

2.4 Symbolic dynamics (entirely optional)

Denote by $\{0,1,2\}^{\mathbb{N}}$ the set of all infinite sequences of the symbols 0, 1, and 2. We think of an element $\omega = \omega_1\omega_2\omega_3\cdots \in \{0,1,2\}^{\mathbb{N}}$ as a word with first letter $\omega_1 \in \{0,1,2\}$, second letter $\omega_2 \in \{0,1,2\}$, and so on. For $\omega, \nu \in \{0,1,2\}^{\mathbb{N}}$, define $d(\omega,\nu) = 0$ if $\omega = \nu$ and otherwise define

$$d(\omega,\nu) = \frac{1}{3^k},$$

where $k \in \mathbb{N}$ is least such that $\omega_k \neq \nu_k$. Define $\sigma : \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ to be the map which forgets the first letter and shifts the remaining letters to the left: $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$.

2.18 Show that d is a metric on $\{0,1,2\}^{\mathbb{N}}$ and that $(\{0,1,2\}^{\mathbb{N}}, d)$ is a compact metric space.

2.19 Show that $\sigma : \{0,1,2\}^{\mathbb{N}} \to \{0,1,2\}^{\mathbb{N}}$ is a continuous surjection of $\{0,1,2\}^{\mathbb{N}}$. Show that σ is not injective, and describe the preimage $\sigma^{-1}(\{\omega\})$.

2.20 Give an example of a point with dense σ orbit in $\{0,1,2\}^{\mathbb{N}}$.

2.21 Choose any previous problem about the map T_3 on $[0,1)$ and reinterpret it to be a problem about the map σ on $\{0,1,2\}^{\mathbb{N}}$.

2.22 Consider the map $\Psi : \{0,1,2\}^{\mathbb{N}} \to [0,1]$ defined by

$$\Psi(\omega) = \sum_{n=1}^{\infty} \frac{\omega_n}{3^n}.$$

Show that Ψ is continuous, surjective, and nearly 1–1.

2.23 Describe the pre-images of the fundamental intervals under Ψ in $\{0,1,2\}^{\mathbb{N}}$. (Such sets are examples of so-called **cylinder sets**.)

2.24 Show that the map Ψ intertwines the maps σ and T_3 in the sense that

$$\Psi \circ \sigma = T_3 \circ \Psi.$$

(Thus, the dynamical systems $([0,1), T_3)$ and $(\{0,1,2,\}^{\mathbb{N}}, \sigma)$ are **isomorphic**, at least in some sense that can be made precise later on.)

Lesson 2: Continued fractions

1 Material

The goal of Lesson 2 is to introduce continued fractions coming from the Euclidean algorithm and dynamical systems.

1.1 Euclidean algorithm

Let $x \in (0,1)$, then we can write

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_1, a_2, a_3, \dots] \tag{1}$$

where the a_k are positive integers. This expansion is finite if and only if x is rational. One direction comes from collapsing the continued fraction expansion into the standard fractional representation, and the other follows from the Euclidean algorithm.

Theorem 1. *Let $a, b \in \mathbb{Z}$ with $a \geq b > 0$. By the division algorithm, there exist unique $q_1, r_1 \in \mathbb{Z}$ such that*

$$a = bq_1 + r_1, \quad 0 \leq r_1 < b.$$

If $r_1 > 0$, (by the division algorithm) there exist unique $q_2, r_2 \in \mathbb{Z}$ such that

$$b = r_1 q_2 + r_2, \quad 0 \leq r_2 < r_1.$$

If $r_2 > 0$, (by the division algorithm) there exist unique $q_3, r_3 \in \mathbb{Z}$ such that

$$r_1 = r_2 q_3 + r_3, \quad 0 \leq r_3 < r_2.$$

Continuing this process, $r_n = 0$ for some n. If $n > 1$, then $(a,b) = r_{n-1}$. If $n = 1$, then $(a,b) = b$.

We can use the Euclidean algorithm to find the continued fraction expansion of $\dfrac{a}{b}$ for integers $a > b > 0$. First, we let $r_0 = a$ and $r_1 = b$.

$$a = r_0 = r_1 a_1 + r_2 \quad 0 \leq r_2 < r_1 = b$$

If $0 = r_2$, we stop. Otherwise,

$$b = r_1 = r_2 a_2 + r_3 \quad 0 \leq r_3 < r_2$$

Continuing until $0 = r_{n+1} < r_n < r_{n-1} < r_{n-2} < \cdots < r_1 = b < r_0 = a$. We know that $r_k = a_{k+1} r_{k+1} + r_{k+2}$ for $k \leq n-1$ and $(a,b) = r_n$. Also, $\frac{a}{b} > 1$, so we are going to find the continued fraction expansion of $x = \frac{b}{a}$.

Then

$$\begin{aligned}
\frac{1}{x} = \frac{a}{b} = \frac{r_0}{r_1} &= a_1 + T_1, & a_1 &= \left\lfloor \frac{r_0}{r_1} \right\rfloor, T_1 = \frac{r_2}{r_1} \\
\frac{1}{T_1} &= a_2 + T_2, & a_2 &= \left\lfloor \frac{r_1}{r_2} \right\rfloor, T_2 = \frac{r_3}{r_2} \\
&\vdots \\
\frac{1}{T_{n-2}} &= a_{n-1} + T_{n-1}, & a_{n-1} &= \left\lfloor \frac{r_{n-2}}{r_{n-1}} \right\rfloor, T_{n-1} = \frac{r_n}{r_{n-1}} \\
\frac{1}{T_{n-1}} &= a_n + 0 & a_n &= \left\lfloor \frac{r_{n-1}}{r_n} \right\rfloor.
\end{aligned}$$

Substituting into x, we find

$$x = \frac{1}{a_1 + T_1} = \frac{1}{a_1 + \cfrac{1}{a_2 + T_2}} = \cdots = \frac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} \tag{2}$$

Example 2. Determine the continued fraction expansion for $\frac{13}{36}$.

First, we apply the Euclidean algorithm to $36 = r_0$ and $13 = r_1$.

$$\begin{aligned}
36 &= 13 * 2 + 10, & a_1 &= 2, r_2 = 10 \\
13 &= 10 * 1 + 3, & a_2 &= 1, r_3 = 3 \\
13 &= 3 * 3 + 1, & a_3 &= 3, r_4 = 1 \\
3 &= 3 * 1 + 0, a_4 & &= 3, r_5 = 0.
\end{aligned}$$

Then $x = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3}}}}$.

Note that finite continued fraction expansions are not unique since $n = n - 1 + \frac{1}{1}$.

1.2 Digits and dynamics

Example 3. Now we do an example of an irrational number, the golden ration $G = \varphi = \frac{1+\sqrt{5}}{2}$. Now, G is a root of $x^2 - x - 1$, so $G = 1 + \frac{1}{G}$. Substituting in for G, we get $G = 1 + \cfrac{1}{1 + \cfrac{1}{G}} = [1; \overline{1}]$ where the $\bar{\cdot}$ indicates repeating digits, as with decimal expansions.

Since every digit is 1, this continued fraction expansion converges very slowly.

Most of the time, we cannot find a nice equation to solve giving the irrational numbers this way, so we need a dynamical system like we saw with ternary numbers. As with ternary numbers, the first digit of the continued fraction tells us which of the fundamental intervals

$$I_1 = \left(\frac{1}{2}, 1\right], I_2 = \left(\frac{1}{3}, \frac{1}{2}\right], \ldots, I_k = \left(\frac{1}{k+1}, \frac{1}{k}\right], \ldots$$

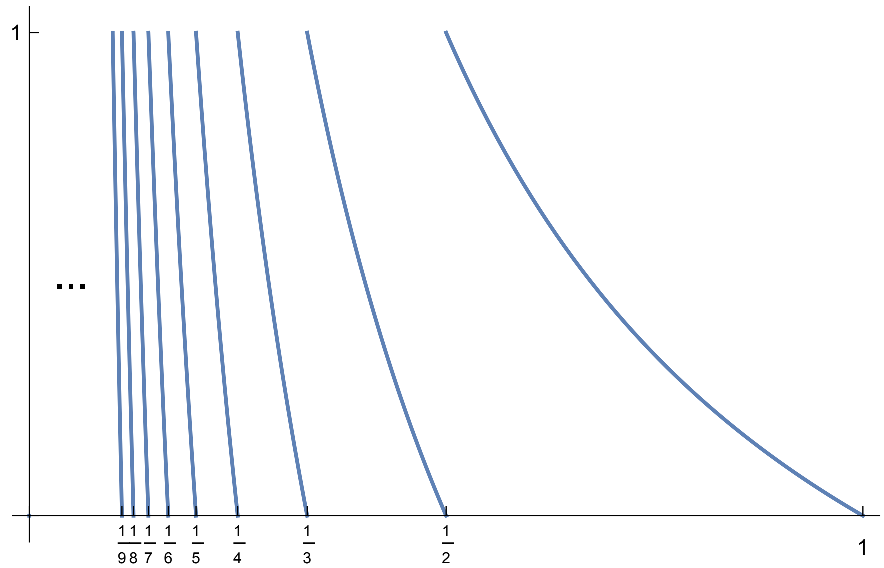
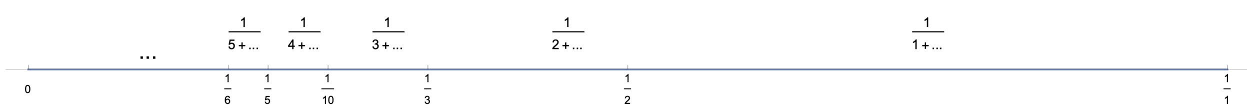


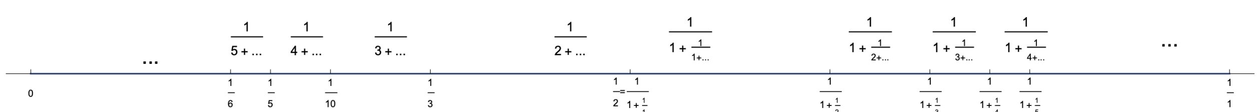
Figure 5: The first 9 branches of the Gauss map.

contains x.



Each fundamental interval is further divided to find the second digit. For example, I_1 is divided into

$$I_{1,1} = \left(\frac{1}{1+\frac{1}{1}}, \frac{1}{1+\frac{1}{2}} \right), I_{1,2} = \left[\frac{1}{1+\frac{1}{2}}, \frac{1}{\frac{1}{3}} \right), \ldots, I_{1,k} = \left[\frac{1}{1+\frac{1}{k}}, \frac{1}{1+\frac{1}{k+1}} \right), \ldots$$



As with the ternary numbers, we find a point $x \in [0,1]$ can have two different regular continued fraction expansions if and only if it lies at the endpoints of one of the fundamental intervals. Now we get that one expansion ends in n and the other ends in $n-1+\frac{1}{1}$. We will see that this means the end points of the fundamental intervals are exactly the rational numbers.

Definition 4. The **regular continued fraction** (or **Gauss**) **map** $T : [0,1) \to [0,1)$ is

$$Tx := \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } x = 0, \end{cases} \tag{3}$$

as in Figure 1.2.

As we saw with the fundamental intervals, $x \in \left(\frac{1}{k+1}, \frac{1}{k} \right]$ means that $a_1(x) = k$. Thus, $Tx = \frac{1}{x} - a_1$ and for $x = [a_1, a_2, a_3, \ldots], T[a_1, a_2, a_3, \ldots] = [a_2, a_3, a_4, \ldots]$. We can use this property to define the continued fraction expansion of any $x \in [0,1)$

$$a_n = a_n(x) = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor, n \geq 1.$$

Definition 5. Let $x = [a_0; a_1, a_2, \dots]$. We call the rational approximations $\dfrac{p_i}{q_i} = [a_0; a_1, a_2, \dots, a_i]$ are called *convergents*, where $(p_i, q_i) = 1$.

We can ask the same questions that we asked with ternary numbers:

(I) Which points $x \in X$, if any, are **fixed points** of T: $Tx = x$? More generally, which are **periodic**: there exists $1 \leq n$ such that $T^n x = x$? Even more generally, which are **eventually periodic**: there exist $1 \leq m < n$ such that $T^n x = T^m x$? If there are any such points, how many are there? Is there an effective way to find them all? Is there a nice classification of such points? (see Lesson 2 problem set)

(II) Which points $x \in X$ exhibit **recurrence**: for any open interval I containing x, there exists $n \in \mathbb{N}$ such that $T^n x \in I$? If a point recurs, how often?

(III) Which points $x \in X$ have a **dense orbit**: the set $\{x, Tx, T^2 x, \dots\}$ is dense in X? Does **almost every** point have a dense orbit: does the set of points without a dense orbit have measure zero (see the Lesson 1 problem set for T_3)?

(IV) How much time does a point $x \in X$ spend in the interval $[1/3, 1/2]$: what proportion of $0 \leq n \leq N$ are such that $T^n x \in [1/3, 1/2]$? Is the sequence $n \mapsto T^n\big(1/\sqrt{2}\big)$ **equidistributed**: does the orbit of $1/\sqrt{2}$ spend $(b-a)$ proportion of the time in the interval $[a, b]$ for every $0 \leq a < b < 1$?

(V) How close do two points $x, y \in X$ come to one another along their orbits: what is $\inf_{n \in \mathbb{N}} |T^n x - T^n y|$? Does the point $(x, y) \in X \times X$ have dense orbit under $T \times T$: is the set $\{(x, y), (Tx, Ty), (T^2 x, T^2 y) \dots\}$ dense in $X \times X$?

2 Problems

2.1 Finding continued fraction expansions

2.1 ([DK] Exercise 1.3.1) Use the Euclidean algorithm to find $(13, 20), (5, 14)$, and $(171951, 31250000)$.

2.2 ([DK] Exercise 1.3.2) Find the continued fraction expansion of $\frac{13}{20}, \frac{5}{14}$, and $\frac{171951}{31250000}$. Compare the number of digits in the continued fraction expansions to the decimal expansions.

2.3 Find the continued fraction expansion of $\sqrt{2} - 1$.

2.4 Verify that for $x = [a_1, a_2, \dots], \frac{1}{x} = a_1 + [a_2, a_3, \dots] = [a_0; a_1, a_2, \dots]$, so we can actually find the continued fraction expansion of any positive real number.

2.2 Continued fraction and Gauss map properties

2.5 ([EW] Exercise 3.1.1) Show that any positive rational number has exactly two continued fraction expansions, and both are finite.

For this section, $T = \frac{1}{x} \mod 1$ as in 1.2

2.6 Classify the fixed points $x \in [0, 1)$ where $Tx = x$.

2.7 Give an example of a point with an eventually periodic orbit under T of period 7.

2.8 Show that any real irrational number x with an eventually periodic continued fraction expansion is the root of a quadratic polynomial.

2.9 Show that the set of periodic points of T are dense in $[0, 1]$.

2.10 Give an example of a real number whose orbit avoids the interval $[0.36, 0.4]$.

2.11 Give an example of a real number with a dense orbit under T. Conclude (immediately) that the set of points with a dense orbit under T is dense.

2.3 Matrix representation of continued fractions

Note: whatever problems we are unable to finish today will move to Lesson 7 on Thursday morning.

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{Z}, ad-bc = \pm 1\}$. We define the **Möbius transformation** as a map $A : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ by

$$Ax := \frac{ax+b}{cx+d}. \tag{1}$$

Definition 6. Let $x = [a_0; a_1, a_2, \ldots]$. For $n \geq 1$, define A_n as

$$\begin{aligned} A_0 := \left(\begin{smallmatrix} 1 & a_0 \\ 0 & 1 \end{smallmatrix}\right), A_n := \left(\begin{smallmatrix} 0 & 1 \\ 1 & a_n \end{smallmatrix}\right) \\ M_n := A_0 A_1 \cdots A_n. \end{aligned} \tag{2}$$

2.12 ([DK] Exercise 1.3.6) Show that if $A, B \in \mathrm{GL}(2,\mathbb{Z})$, then $(AB)x = A(Bx)$.

2.13 ([DK] Exercise 1.3.7) Considering the matrices A_n and M_n as Möbius transformations, show that

$$M_n 0 = [a_0; a_1, a_2, \ldots, a_n].$$

and

$$A_1 A_2 \ldots A_n = (A_n A_{n-1} \ldots A_1)^T.$$

2.14 ([DK] Exercise 1.3.8) Writing $M_n = \left(\begin{smallmatrix} r_n & p_n \\ s_n & q_n \end{smallmatrix}\right), n \geq 0$, show:

(a) $(p_n, q_n) = 1$,

(b) $r_n = p_{n-1}, s_n = q_{n-1}$,

(c) $p_{n-1}q_n - p_n q_{n-1} = (-1)^n, n \geq 1$,

(d) $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$ as above,

(e) the sequences $(p_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ satisfy the following recurrence relations

$$\begin{aligned} p_{-1} &= 1, & p_0 &= a_0, p_n = a_n p_{n-1} + p_{n-2} \\ q_{-1} &= 0, & q_0 &= 1, q_n = a_n q_{n-1} + q_{n-2} \end{aligned}$$

(f) $p_n(x) = q_{n-1}(Tx)$ for all $n \geq 1$.

(g) the sequence $(q_n)_{n\geq 1}$ is a monotone increasing sequence of positive integers.

2.15 ([DK] Exercise 1.3.9) Use the recurrence relation for q_n to show

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_1].$$

These next few problems will prove that $\lim_{n\to\infty} \frac{p_n}{q_n} = x$. We define one more matrix

$$A_n^* := \begin{pmatrix} 0 & 1 \\ 1 & a_n + T_n \end{pmatrix}, n \geq 1, T_n = T^n x.$$

2.16 ([DK] Exercise 1.3.10)

(a) Show that $x = (M_{n-1}A_n^*)0$.

(b) Use the the fact that

$$M_{n-1} = \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix}, n \geq 1$$

and the recurrence relations for $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ to show that

$$x = \frac{p_n + T_n p_{n-1}}{q_n + T_n q_{n-1}}, \quad n \geq 1$$

ie, $x = m_n(T^n x)$.

(c) Use the fact that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ to conclude that

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T_n}{q_n(q_n + T_n q_{n-1})}, \quad n \geq 1.$$

(d) Since $T_n \in [0, 1)$ we have that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}, \quad n \geq 1.$$

(e) $\lim_{n\to\infty} \frac{p_n}{q_n} = x$.

2.17 Here is an alternate proof of the fact that $\lim_{n\to\infty} \frac{p_n}{q_n} = x$. Use the fact that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ to conclude that

$$\frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

and thus $\lim_{n\to\infty} \frac{p_n}{q_n} = x$.

2.18 ([DK] Exercise 1.3.11) Show that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Define **cylinder set**

$$\Delta_n = \Delta(a_1, a_2, \ldots, a_n) := \{x \in (0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

The cylinder sets are fundamental intervals of rank n for the Gauss map.

2.19 ([DK] Exercise 1.3.13) Show that $\Delta(1) = \left(\frac{1}{2}, 1\right)$ and $\Delta(n) = \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \geq 2$. Determine $\Delta(1, 1)$ and $\Delta(m, n)$ for $m, n \geq 1$.

2.20 ([DK] Exercise 1.3.15) Show that $\Delta(a_1, a_2, \ldots, a_k)$ is an interval in $[0, 1)$ with endpoints

$$\frac{p_k}{q_k} \quad \text{and} \quad \frac{p_k + p_{k+1}}{q_k + q_{k-1}}.$$

Lesson 3: Frequencies and measures

The purpose of this lesson is to introduce the ergodic theory of interval maps, define absolutely continuous invariant measures, and discuss what it means for a set to have measure zero.

1 Material

1.1 What is ergodic theory?

Dynamical systems starts with the study of **iterations** of a map $T : X \to X$. It asks questions about the sequence

$$T(x), T(T(x)), T(T(T(x))), \ldots, T^n(x), \ldots \tag{1}$$

for points $x \in X$. We write $T^n(x)$ for the n-fold composition of T with itself so that

$$T^7(x) = T(T(T(T(T(T(T(x)))))))$$

and so on. The sequence (1) is the **orbit** of x under T. The structure of X and the properties of T influence which questions one asks and how well they can be answered. For example, if X is a metric space then one can ask about accummulation points of orbits, and if T is smooth then one can ask about the derivatives of T^n etc.

This course is about **ergodic theory**. This is the part of dynamical systems dealing with statistical questions about orbits. The historical roots of ergodic theory are in statistical mechanics, specifically in the kinetic theory of gasses, where one studies a large number of gas molecules in a confined region undergoing elastic collisions. In this course we will focus on the situation where $X = [0, 1)$ and $T : [0, 1) \to [0, 1)$ is piecewise smooth. Although this may seem restrictive, it covers a lot of important examples and lets us avoid some measure theory.

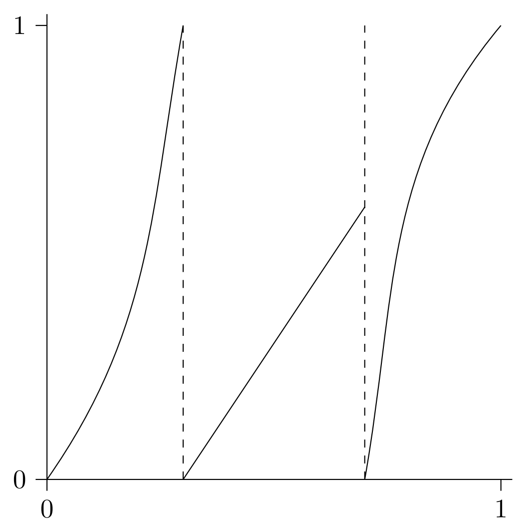


Figure 6: A cartoon of a piecewise-smooth map $T : [0, 1) \to [0, 1)$.

One of the simplest observations we can record from the orbit (1) of a point x is those iterates n for which $T^n(x)$ belongs to a specific set $A \subset [0, 1)$. It is natural to take the average of our observations: to consider the frequency

$$\frac{|\{1 \le n \le N : T^n(x) \in A\}|}{N}$$

of visits, up to N iterates, of our point's orbit to the set A. For example, if $T(x) = 3x \bmod 1$ and A is the interval $[\frac{1}{3}, \frac{2}{3})$ then $|\{1 \le n \le N : T^n(x) \in A\}|$ is the number of occurrences of 1 amongst the ternary digits $d_2(x), \ldots, d_{N+1}(x)$ of x.

1.2 Ergodic averages

The fundamental question we wish to address is whether observed frequencies have a limiting behaviour. In other words, does the limit

$$\nu_x(A) = \lim_{N\to\infty} \frac{|\{1 \le n \le N : T^n(x) \in A\}|}{N} \tag{2}$$

exist for a given point $x \in X$ and a given set $A \subset X$?

Suppose for a moment that the limit (2) is known to exist for a certain point $x \in [0,1)$ and some set $A \subset [0,1)$. If we highlight in red the terms in the orbit of x that belong to A then the limit is the asymptotic proportion of red terms in the sequence

$$\color{red}{T(x)} \quad T^2(x) \quad T^3(x) \quad \color{red}{T^4(x)} \quad \color{red}{T^5(x)} \quad T^6(x) \quad \color{red}{T^7(x)} \quad T^8(x) \quad T^9(x) \quad \color{red}{T^{10}(x)} \quad T^{11}(x) \quad T^{12}(x) \quad \cdots$$

One can verify that it is equal to the proportion of red terms in the orbit of $T(x)$

$$T^2(x) \quad T^3(x) \quad \color{red}{T^4(x)} \quad \color{red}{T^5(x)} \quad T^6(x) \quad \color{red}{T^7(x)} \quad T^8(x) \quad T^9(x) \quad \color{red}{T^{10}(x)} \quad T^{11}(x) \quad T^{12}(x) \quad \color{red}{T^{13}(x)} \quad \cdots$$

Thus

$$\nu_x(A) = \nu_{T(x)}(A) = \nu_x(T^{-1}A) \tag{3}$$

where $T^{-1}A = \{x \in X : T(x) \in A\}$ is the set of points that T will map into A.

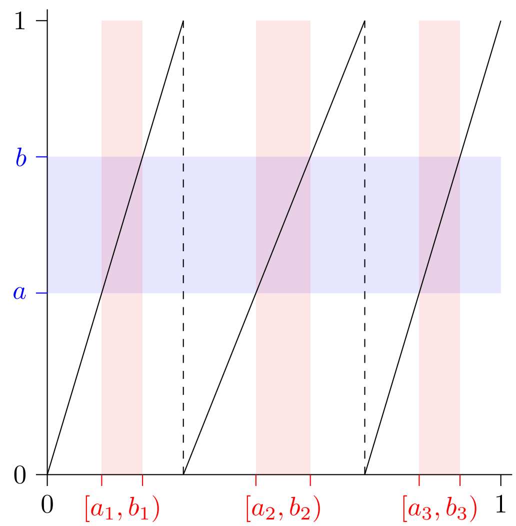


Figure 7: A cartoon of $T^{-1}([a,b)) = [a_1,b_1) \cup [a_2,b_2) \cup [a_3,b_3)$ for a piecewise-smooth map $T : [0,1) \to [0,1)$.

Of course we do not – and should not easily expect to – know that the limit (2) actually exists. But whether it does or not, one can still take up the study of maps ν assigning numbers $0 \le \nu(A) \le 1$ to sets $A \subset [0,1)$ in such a way that $\nu(T^{-1}(A)) = \nu(A)$. A full understanding of such maps – and what their domains may be – requires the introduction of measure theory. We do not wish to assume, or dwell on, the details of measure theory this week. We will therefore only consider measures defined by densities, brushing some details aside. Given a function $\phi : [0,1) \to [0,\infty)$ one can define

$$\nu(A) = \int_A \phi(x)\,\mathsf{d}x \tag{4}$$

for sets $A \subset [0,1)$. A map of subsets of $[0,1)$ defined by (4) is called an **absolutely continuous measure** on $[0,1)$. If one takes $\phi = 1$ then the resulting measure is called **Lebesgue measure**.

In fact, for (4) to make sense we must assume something ("measurability") of ϕ and must content ourselves with inputs $A \subset [0,1)$ belonging to a large ("Lebesgue measurable") but not exhaustive class of subsets; these are the measure theory issues we wish to elide. For the rest of the week the map ϕ will usually be continuous and A will often be an interval, so that (4) can often be thought of as a Riemann integral and handled using calculus techniques.

1.3 Absolutely continuous invariant measures

A function $\phi : [0,1) \to [0,\infty)$ is an **invariant density** for a map $T : [0,1) \to [0,1)$ if

- $\displaystyle\int\limits_{T^{-1}([a,b))} \phi(x)\,\mathsf{d}x = \int\limits_{[a,b)} \phi(x)\,\mathsf{d}x$ for all $0 \le a < b \le 1$.

- $\displaystyle\int\limits_0^1 \phi(x)\,\mathsf{d}x = 1$

both hold. In other words ϕ is an invariant density for T when the measure ν defined by (4) satisfies $\nu(T^{-1}(A)) = \nu(A)$ for $A \subset [0,1)$ and $\nu([0,1)) = 1$. We do in fact get $\nu(T^{-1}(A)) = \nu(A)$ for all (measurable) sets $A \subset [0,1)$ even though the first requirement above only refers to intervals. When T is piecewise smooth the set $T^{-1}([a,b))$ is a disjoint union of intervals $[a_1,b_1),[a_2,b_2),\ldots$ and

$$\int\limits_{T^{-1}([a,b))} \phi(x)\,\mathsf{d}x = \int\limits_{[a_1,b_1)} \phi(x)\,\mathsf{d}x + \int\limits_{[a_2,b_2)} \phi(x)\,\mathsf{d}x + \cdots$$

is again a sum of integrals over intervals, so checking the first condition does not involve sets more complicated than intervals.

We will see many examples of invariant densities in the problem sets. For now let us record that $\phi = 1$ is an invariant density for the ternary map $T(x) = 3x \bmod 1$. In other words, Lebesgue measure is an absolutely continuous invariant measure for the ternary map.

An absolutely continuous measure ν with a density ϕ that is invariant for a map $T : [0,1) \to [0,1)$ is called an **absolutely continuous invariant measure** for T. With an absolutely continuous invariant measure ν to hand, we can now return to our frequency question, fixing an interval $[a,b) \subset [0,1)$ and asking whether there are *any* points $x \in [0,1)$ for which

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N : T^n(x) \in [a,b)\}|}{N} = \nu([a,b)) \tag{5}$$

holds.

Such a result would be quite remarkable. It would mean that one could divine the average outcome of the experiment in which one records the visits of the orbit of x to the interval $[a,b)$ without having to perform any measurements!

Unfortunately, it is not possible in general for (5) to hold for *all* points $x \in [0,1)$. Indeed, we have already seen in the exercises that the middle thirds Cantor set C is invariant under the ternary map T so that (5) will *not* hold when $x \in C$ and $[a,b)$ is disjoint from C.

Amazingly enough, however, it will turn out – under assumptions that are theoretically mild but often difficult to verify in practice – that (5) will hold for most of the points in $[0,1)$.

What does "most" mean? It cannot refer simply to cardinality, as $[0,1)$ and C are of the same size from that point of view. We therefore wish "most" to allow for a set of exceptions that is small in some other sense.

We will end this lesson with a discussion about what it means for a set $A \subset [0,1)$ to be "small". If the complement of the set of points where a proposition – such as (5) – holds is "small" then the proposition is said to hold for "most" points. Formally, a set $A \subseteq [0,1]$ has **measure zero** if, for all $\epsilon > 0$, there exists a countable (finite or infinite) collection $\{I_n\}_{n=1}^{\infty}$ of (open, closed, or half-open) intervals in $[0,1]$ such that

- the intervals cover X in that $X \subseteq \displaystyle\bigcup_{n=1}^{\infty} I_n$;

- the total length of the intervals is at most ϵ i.e. $\displaystyle\sum_{n=1}^{\infty} \text{Length}(I_n) < \epsilon$;

where the **length** of an interval $[a, b)$ is just $b - a$. We say that a property of points $x \in [0, 1)$ holds **almost surely** if the set of points for which the property *does not* hold is of measure zero. For example, as we will see in the problems, almost every $x \in [0, 1)$ is irrational, and almost every $x \in [0, 1)$ has a 1 in its ternary expansion i.e. almost every point $x \in [0, 1)$ is outside Cantor's middle thirds set.

2 Problems

2.1 Frequencies in dynamical systems

2.1 Let $X = \{z \in \mathbb{C} : z^5 = 1\}$ be the set of fifth roots of unity. Define $T : X \to X$ by $T(z) = e^{2\pi i/5} z$ for all $z \in X$.

(a) Describe the set $\{n \in \mathbb{N} : T^n(1) \in \{e^{2\pi i/5}\}\}$.

(b) Describe the set $\{n \in \mathbb{N} : T^n(1) \in \{1, e^{6\pi i/5}\}\}$.

(c) Compute the limit

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in \{1\}\}|}{N}$$

for each $x \in X$.

2.2 Let $X = \mathbb{R}^2$. Put

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and define $T : X \to X$ by $T(x) = Ax$ for all $x \in \mathbb{R}^2$.

(a) Describe the set $\{T^n(1 + \sqrt{5}, 2) : n \in \mathbb{N}\}$.

(b) Describe the set $\{T^n(1 - \sqrt{5}, 2) : n \in \mathbb{N}\}$.

(c) Verify that $\{(1 + \sqrt{5}, 2), (1 - \sqrt{5}, 2)\}$ is a basis of $\mathbb{R}^2$.

(d) What happens to the sequence $T^n(x)$ as $n \to \infty$? (There are – arguably – four different cases for $x \in X$.)

(e) Fix $A \subset X$ closed and bounded with $0 \notin A$. Compute the limit

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in A\}|}{N}$$

for each $x \in X$.

2.3 Let $T(x) = 3x \bmod 1$ and $A = [\frac{1}{3}, \frac{2}{3})$.

(a) Give an example of a number $x \in [0, 1)$ for which the limit $\nu_x(A)$ equals $\frac{1}{4}$.

(b) How might you try to construct a number $x \in [0, 1)$ for which the limit $\nu_x(A)$ equals $\frac{1}{\pi}$?

(c) Does the limit $\nu_x(A)$ exist for every $x \in [0, 1)$?

2.4 Fix a bounded sequence a_n of real numbers.

(a) Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n - \frac{1}{N} \sum_{n=1}^{N} a_{n+k} = 0$$

for every $k \in \mathbb{N}$.

(b) Fix $T : [0, 1) \to [0, 1)$ and $x \in [0, 1)$ and $A \subset [0, 1)$ such that the limit $\nu_x(A)$ exists. Verify that $\nu_x(A) = \nu_{T(x)}(A) = \nu(T^{-1}A)$.

2.2 Absolutely continuous invariant measures

Fix an absolutely continuous measure ν defined by

$$\nu(A) = \int_A \phi(x)\,\mathsf{d}x$$

for sets $A \subset [0, 1)$.

2.5 Take for granted that if $A, B \subset [0, 1)$ are disjoint then $\nu(A \cup B) = \nu(A) + \nu(B)$.

(a) Prove that $\nu(A) \le \nu(B)$ whenever $A \subset B$.

(b) Prove by induction that
$$\nu(A_1 \cup \cdots \cup A_d) = \nu(A_1) + \cdots + \nu(A_d)$$
whenever $A_1, \ldots, A_d \subset [0, 1)$ satisfy $A_i \cap A_j = \emptyset$ for all $1 \le i \ne j \le d$.

(c) Prove that if $\nu(A) > 0$ then for every $k \in \mathbb{N}$ there is $0 \le i < 2^k$ with $\mu(A \cap [\frac{i}{2^k}, \frac{i+1}{2^k})) \ge \nu(A)/2^k$.

2.6 Verify that $\phi(x) = 1$ is an invariant density for the ternary map $T : [0, 1) \to [0, 1)$ defined by $T(x) = 3x \bmod 1$.

2.7 Fix $\alpha \in \mathbb{R}$. Verify that $\phi(x) = 1$ is an invariant density for the rotation map $T : [0, 1) \to [0, 1)$ defined by $T(x) = x + \alpha \bmod 1$.

2.8 Verify that
$$\phi(x) = \frac{1}{\log 2} \frac{1}{1 + x}$$
is an invariant density for the Gauss map $T : [0, 1) \to [0, 1)$ defined by $T(x) = \frac{1}{x} \bmod 1$.

***2.9** Prove that the map $T : [0, 1) \to [0, 1)$ defined by $T(x) = \sqrt{x}$ does not have an invariant density.

***2.10** Fix $\beta > 1$ and define $T : [0, 1) \to [0, 1)$ by $T(x) = \beta x \bmod 1$. Prove that
$$h_\beta(x) = \sum_{\substack{n \ge 0 \\ x < T^n(1)}} \frac{1}{\beta^n}$$
is an invariant density for T.

2.3 Sets of measure zero

2.11 Prove that the singleton $\{x\}$ has zero measure for every $x \in [0, 1)$.

2.12 Prove that the union of two sets of measure zero has measure zero, and conclude that every finite subset of $[0, 1]$ has measure zero.

2.13 Prove that a countable union of sets of measure zero has measure zero, and conclude that $\mathbb{Q} \cap [0, 1]$ has measure zero.

***2.14** Prove that $[0, 1]$ does not have measure 0.

2.15 Prove that Cantor's middle thirds set C has measure zero.

2.16 Prove that the set of real numbers in $[0, 1)$ whose ternary expansion avoids two consecutive 1s has measure zero. (How is that set related to C?)

Interval exchanges Fix a permutation σ of $\{1, \ldots, d\}$ and positive numbers $\lambda_1, \ldots, \lambda_d$ that sum to 1. Let $I_1, \ldots, I_d$ be the intervals $[0, \lambda_1), \ldots, [\lambda_1 + \lambda_{d-1}, \lambda_1 + \lambda_d)$ respectively. Define $T : [0, 1) \to [0, 1)$ by

$$T(x) = x - \sum_{j<i} \lambda_j + \sum_{\pi(j)<\pi(i)} \lambda_j$$

for all $x \in I_i$. Any such T is called an **interval exchange transformation**.

2.17 What is T when $d = 2$ and $\pi = (1, 2)$?

2.18 Prove that every interval exchange transformation has $\phi = 1$ as an invariant density.

Lesson 4: Continued fractions and the hyperbolic plane

1 Material

The lesson goes through the connection between continue fractions and the hyperbolic plane. Some of the geometric facts will be stated without proof. The lecture portion of this lesson will be an overview of the results, while the problems work through some of the proofs touching on other areas of ergodic theory. This lesson is largely based on Caroline Series' paper [Ser].

1.1 Hyberbolic plane and two dimensional transformations

We start with the **upper half plane** $\mathbb{H} := \{x + iy \mid y > 0\} \cup \{\infty\}$ where the shortest path between two points lies on a semicircle centered on the real (or $x-$) axis or is the vertical line $x+i\mathbb{R}$. We call the semicircle (or line) a **geodesic** and points where the geodesic intersects the the **endpoints of the geodesic**. One parameterization of a geodesic is $\gamma(t) = \frac{aie^t+b}{ie^t+1}$. Here, $\lim_{t\to\infty}\gamma(t) = a$ and $\lim_{t\to-\infty}\gamma(t) = b$, so we call a the **forward endpoint** and b the **backwards endpoint**.

For this construction, we consider the Möbius transformations $M : \mathbb{H} \to \mathbb{H}, Mz = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ where $M \in \mathrm{PSL}(2,\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}/\{\pm I\}$. Notice that modding out by $\{\pm I\}$ accounts for the fact that $\frac{-az-b}{-cz-d} = \frac{az+b}{cz+d}$. We call the set of all such transformation the $\mathrm{PSL}(2,\mathbb{Z})$ **action**. The image of $i\mathbb{R}$ under the $\mathrm{PSL}(2,\mathbb{Z})$ action gives the **Farey tessellation**.

An equivalent definition is start by drawing a vertical geodesic $n + i\mathbb{R}$ at each integer n. Then connect each n to the integers $n-1$ and $n+1$ by a geodesic. Next, connect each n to the $n+\frac{1}{2}$ and $n-\frac{1}{2}$ on either side. In general, two rational numbers are connected by a geodesic if they are adjacent in some (generalized) Farey sequence $F_n = \{\frac{p}{q} : 0 \le q \le n, (p,q) = 1\}$ ordered from smallest to largest. Here, $\infty = \frac{1}{0}$. The first few generalized Farey sequences are:

$$
\begin{aligned}
F_1 &= \left\{\dots, -\frac{1}{1}, \frac{0}{1}, \frac{1}{1}, \dots\right\},\\
F_2 &= \left\{\dots, -\frac{1}{1}, -\frac{1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \dots\right\},\\
F_3 &= \left\{\dots, -\frac{1}{1}, -\frac{2}{3}, -\frac{1}{2} - \frac{1}{3}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \dots\right\},\\
F_4 &= \left\{\dots, -\frac{1}{1}, -\frac{3}{4}, -\frac{2}{3}, -\frac{1}{2} - \frac{1}{3}, -\frac{1}{4}\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \dots\right\}.
\end{aligned}
$$

We will see Farey sequences again in Lesson 8.

We will use how the Farey tessellation cuts a geodesic γ with endpoints $\gamma_{+\infty}$ and $\gamma_{-\infty}$ to find the continued fraction expansions of $\gamma_{+\infty}$ and $\gamma_{-\infty}$. To do this, we define a two dimensional, invertible extension of the Gauss map. This way, we can analyze both endpoints at the same time. Define $\bar{T} : [0,1)^2 \to [0,1)^2$ by

$$
\bar{T}(x,y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{y+k}\right) & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0,y) & \text{if } x = 0 \end{cases}. \tag{1}
$$

Then on the continued fraction representation, $\bar{T}(([a_1, a_2, \dots], [a_0, a_{-1}, \dots])) = ([a_2, a_3, \dots], [a_1, a_0, a_{-1}, \dots])$.

There's a slightly obvious problem here: we are trying to use a geodesic in $\mathbb{H}$ to find the continued fraction expansions of $(\gamma_{+\infty}, \gamma_{-\infty}) \in \mathbb{R}^2$, but $\overline{T}$ is defined on $[0,1)^2$. We solve this by defining a dictionary between geodesics and the continued fraction expansions of their endpoints and defining a map that "acts like $\overline{T}$."

1.2 Cutting Sequences

In order to prove everything in this section completely, we would need define

$$
\mathcal{M} = \mathrm{PSL}(2,\mathbb{Z}) \backslash \mathbb{H} = \{[z] : z \sim w \text{ if there exists } M \in \mathrm{PSL}(2,\mathbb{Z}), Mz = w\}
$$

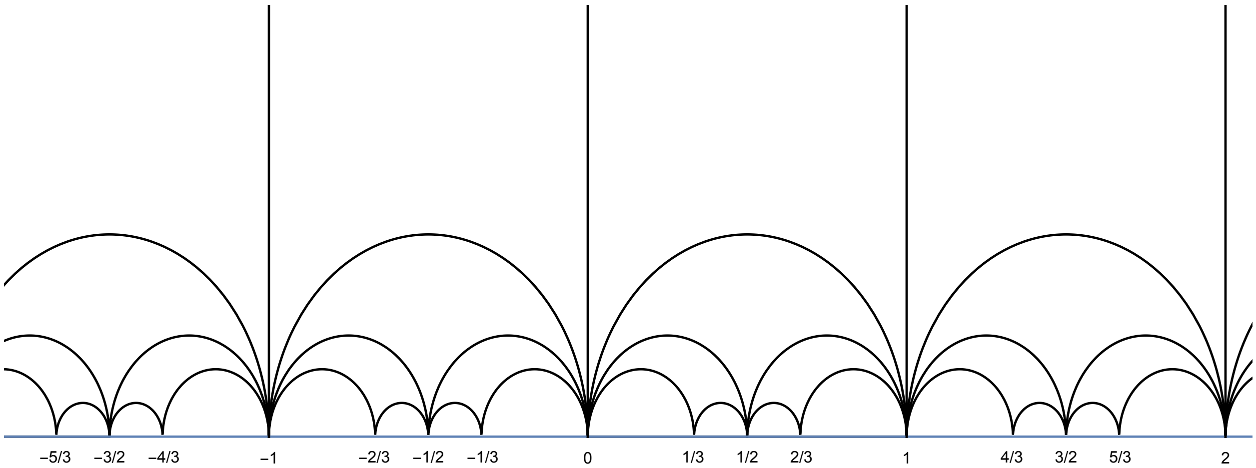


Figure 8: The Farey tessellation up to level 3.

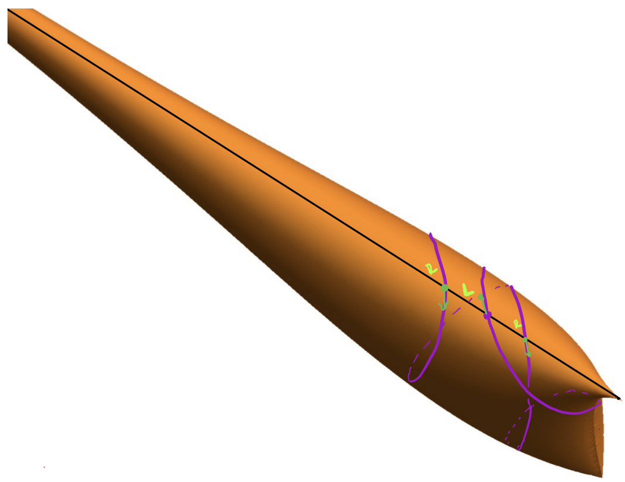


Figure 9: The modular surface $\mathcal{M}$, approximately. The cusp at the top left continues to infinity.

and then work with unit length tangent vectors to the geodesics on $\mathcal{M}$. Figure 9 shows $\mathcal{M}$ with part of a geodesic $\overline{\gamma}$. The black line is the image of $i\mathbb{R}$ under the map $\pi : \mathbb{H} \to \mathcal{M}, \pi(z) = [z]$. The arrows are the unit tangent vectors to $\overline{\gamma}$ based on the black line.

There are a few facts about the hyperbolic plane that allow us to work directly with a subset of geodesics on $\mathbb{H}$:

1.1 Hyperbolic geodesics are unique. We can then identify $(\gamma_{+\infty}, \gamma_{-\infty}) \in \mathbb{R}^2$ with the geodesic γ from $\gamma_{-\infty}$ to $\gamma_{+\infty}$.

1.2 Unit tangent vectors define geodesics. Let ξ_γ be where γ intersects $i\mathbb{R}$ and (u_γ, ξ_γ) the unit tangent vector base at ξ_γ. We can identify (u_γ, ξ_γ) with $(\gamma_{+\infty}, \gamma_{-\infty})$.

1.3 A map on $(\gamma_{+\infty}, \gamma_{-\infty})$ induces a map on (u_γ, ξ_γ).

Thus, in order to define a map on the set of all (u_γ, ξ_γ), we let

$$\mathcal{S} = \{(\gamma_{+\infty}, \gamma_{-\infty}) : |\gamma_{+\infty}| \geq 1, 0 < |\gamma_{+\infty}| < 1, \operatorname{sign}(\gamma_{+\infty}) = -\operatorname{sign}(\gamma_{-\infty})\}$$

and $\mathcal{A} = \{\gamma(t) : (\gamma_{+\infty}, \gamma_{-\infty}) \in \mathcal{S}\}$. That is, $\mathcal{A}$ is the set of oriented geodesics γ with endpoints $\gamma_{-\infty} \in (-1, 0), \gamma_{+\infty} \geq 1$ or $\gamma_{-\infty} \in (0, 1), \gamma_{+\infty} \leq -1$. A map on $\mathcal{S}$ induces a map on $\{(u_\gamma, \xi_\gamma) : \xi_\gamma = iy, y > 1\}$. We will talk about the map $\rho : \mathcal{S} \to \mathcal{S}$, but occasionally need facts about the map on $\{(u_\gamma, \xi_\gamma) : \xi_\gamma = iy, y > 1\}$.

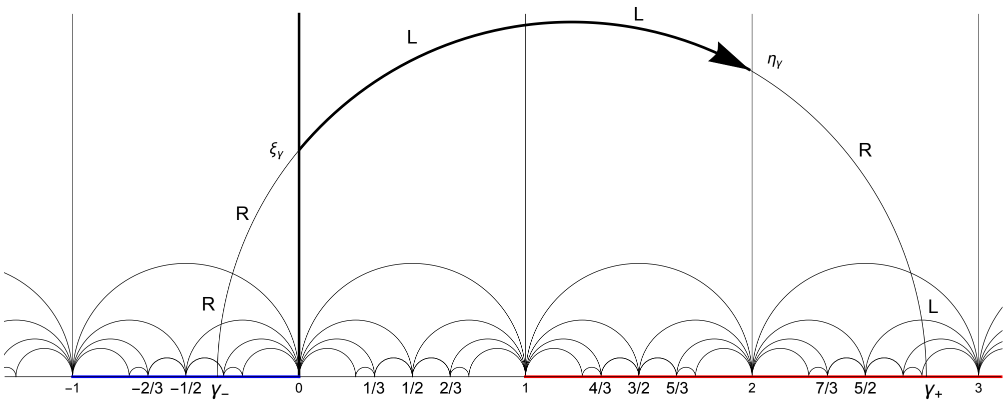


Figure 10: An example of a geodesics with labeled cutting sequence $\ldots RR\xi_\gamma LLRL\ldots$ so $\gamma_{-\infty} = -[0;3,1,\ldots]$ and $\gamma_{+\infty} = [2;1,3,\ldots]$

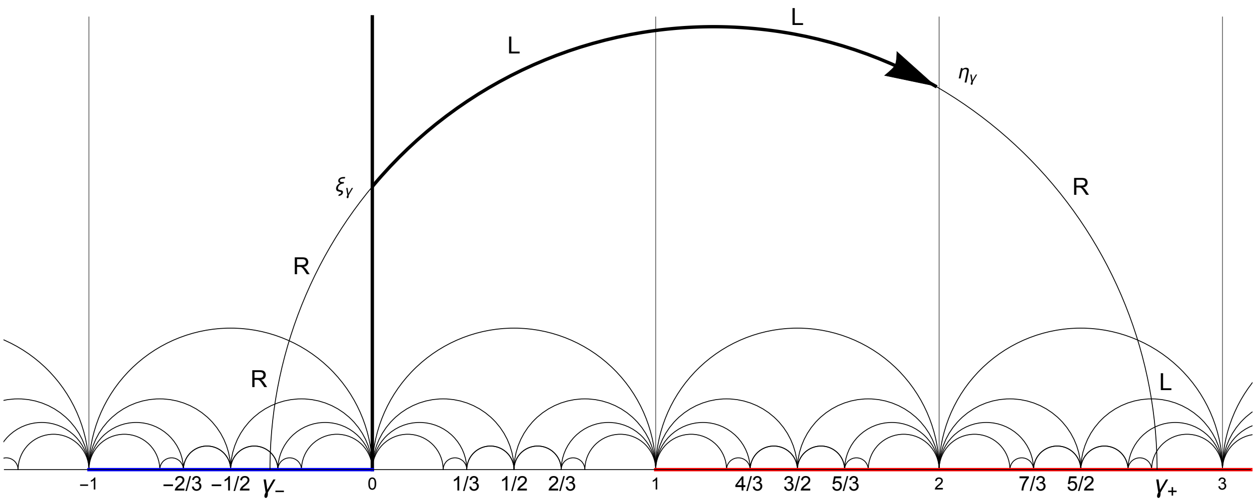


Figure 11: Cutting sequence $\ldots RR\xi L^{n_1}R^1L\ldots$

The Farey tessellation is made of hyperbolic triangles. Each time a geodesic $\gamma(t)$ crosses one of these triangles, it must cross two sides of the triangle. These two sides meet at a vertex that is either on the left of the geodesic (L) or on the right (R). Another way to think about this is the geodesic either turns left when passing through the triangle, or it turns right. Figure 9 shows a geodesics $\overline{\gamma(t)} = \pi(\gamma(t))$ labeled L if the cusp is on the left of the geodesic and R if the cusp is on the right. We call this labeling a **cutting sequence**

Figures 10 and 11 show geodesics with part of the corresponding cutting sequences. Notice that in both cases, the letter immediately before ξ_γ is R and the letter immediately after is L. This is true for all $(\gamma_{+\infty}, \gamma_{-\infty}) \in [1,\infty) \times (-1,0)$. When $(\gamma_{+\infty}, \gamma_{-\infty}) \in (-\infty,-1] \times (0,1)$, the letter immediately before ξ_γ is L and the letter immediately after is R. Thus, we say that the cutting sequence **changes type at** ξ_γ. The next place that the cutting sequence changes type is marked η_γ.

Let $X = \{(u_\gamma, \xi) : \text{cutting sequence change type at } \xi\}$.

Theorem 7 ([Ser] Theorem A)**.** *The map $i : \mathcal{A} \to X, i(\gamma) = \pi((u_\gamma, \xi))$ is surjective, continuous, and open. It is injective except for the two oppositely oriented geodesics joining $+1$ to -1 have the same image.*

A geodesic from $\gamma_{-\infty}$ to $\gamma_{+\infty}$ has two options:

- $\gamma_{-\infty} \in (-1,0)$, $\gamma_{+\infty} \in (1,\infty)$. *This geodesic has the coding* $\ldots L^{n_{-2}}R^{n_{-1}}\xi_g a L^{n_0}R^{n_1}L^{n_2}\ldots$,

$$\gamma_{-\infty} = -[0; n_{-1}, n_{-2}, \ldots] \text{ and } \gamma_{+\infty} = [n_0; n_1, n_2, \ldots]$$

- $\gamma_{-\infty} \in (0,1),\ \gamma_{+\infty} \in (-\infty,-1)$. *This geodesic has the coding* $\dots R^{n_{-2}} L^{n_{-1}} \xi_g a R^{n_0} L^{n_1} R^{n_2} \dots,$

$$\gamma_{-\infty} = [0; n_{-1}, n_{-2}, \dots] \text{ and } \gamma_{+\infty} = -[n_0; n_1, n_2, \dots].$$

1.3 Action on Upper Half Plane

Part of proving Theorem 7 is defining a map that acts as a shift map on the cutting sequence. This is the promised map that "acts like $\overline{T}$". One thing that we will need to prove during the problem session is that the endpoints of a geodesic are determined by the cutting sequence.

Define $\rho : \mathcal{S} \to \mathcal{S}$ by

$$\rho(x,y) = \begin{cases} (\frac{1}{a_1 - x}, \frac{1}{a_1 - y}), & a_1 = \lfloor x \rfloor, \\ (\frac{1}{-a_1 - x}, \frac{1}{-a_1 - y}), & a_1 = \lfloor |x| \rfloor. \end{cases} \tag{2}$$

Looking at the cutting sequence, ρ takes

$$\dots L^{n_{-1}} R^{n_0} \xi_\gamma L^{n_1} R^{n_2} \dots \mapsto L^{n_{-1}} R^{n_0} L^{n_1} \xi_{\rho(\gamma)} R^{n_2} \dots$$

One advantage of this construction, other than giving another connection between matrix groups and continued fractions, is we can use some high powered facts about the set $\{(u_\gamma, \xi_\gamma) : \xi_\gamma = iy, y > 1\}$ (and thus $\mathcal{A}$).

Proposition 8 (Corollary to Series' Theorems B & C)**.** *The map $\rho : \mathcal{S} \to \mathcal{S}$ is invertible, and the diagram*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\rho} & \mathcal{S} \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ (0,1]^2 & \xrightarrow{\bar{T}} & (0,1]^2. \end{array}$$

commutes, where $J : \mathcal{S} \to (0,1]^2$ is the invertible map defined by

$$J(x,y) := \operatorname{sign}(x)(1/x, -y)$$

An invariant density ρ is

$$\frac{d\alpha d\beta}{(\alpha - \beta)^2}.$$

This is one of the facts that we will not prove, as it follows from actually calculating out the induced map on $\{(u_\gamma, \xi_\gamma) : \xi_\gamma = iy, y > 0\}$ We will use this fact to find invariant densities for $\bar{T}$ and T. We will also sue that fact that ρ is what we call **ergodic** to show that $\bar{T}$ and T. A map ρ is *ergodic* with invariant measure ν if for every ν-measurable set A such that $\rho^{-1}A = A$, $\nu(A) = 0$ or 1.

2 Problems

2.1 Completing proofs

2.1 Show that the two definitions of the Farey tessellation are equivalent.

2.2 Show that $\overline{T}$ is invertible (bijective).

2.3 Let $\pi : \mathbb{H} \to \mathcal{M}, \pi(z) = [z]$, and by abuse of notation, we write $\pi(\gamma(t)) = \overline{\gamma(t)}$ for the image of a geodesic under this projection, and $\pi(u_\gamma)$ for the unit tangent vector pointing along $\overline{\gamma(t)}$ based at $\pi(i\mathbb{R})$. Let $i : \mathcal{A} \to X$ by $i(\gamma) = \pi(u_\gamma)$ as in Theorem 7. Prove that i is surjective. Prove that i is injective except that the two oppositely oriented geodesics joining $+1$ to -1 have the same image.

2.4 Prove that a geodesic from $\gamma_{-\infty}$ to $\gamma_{+\infty}$ has two options:

- $\gamma_{-\infty} \in (-1,0),\ \gamma_{+\infty} \in (1,\infty)$. This geodesic has the coding $\dots L^{n_{-2}} R^{n_{-1}} \xi_\gamma L^{n_0} R^{n_1} L^{n_2} \dots,$

$$\gamma_{-\infty} = -[0; n_{-1}, n_{-2}, \dots] \text{ and } \gamma_{+\infty} = [n_0; n_1, n_2, \dots]$$

- $\gamma_{-\infty} \in (0,1),\ \gamma_{+\infty} \in (-\infty,-1)$. This geodesic has the coding $\dots R^{n_{-2}} L^{n_{-1}} \xi_\gamma R^{n_0} L^{n_1} R^{n_2} \dots,$

$$\gamma_{-\infty} = [0; n_{-1}, n_{-2}, \dots] \text{ and } \gamma_{+\infty} = -[n_0; n_1, n_2, \dots].$$

2.5 Explain how this construction relates to the two continued fraction expansions for rational points.

2.6 Show that geodesics with the same cutting sequences coincide.

2.7 Prove Proposition 8.

2.8 Show that $\frac{d\alpha d\beta}{(\alpha-\beta)^2}$ is an invariant density for ρ.

Let $T : X \to X$ with invariant measure μ and $f : X \to Y$. The **pushforward of μ by f** is

$$(f_*\mu)B = \mu(f^{-1}B), \quad B \subset Y.$$

2.9 Show that if μ is an invariant measure for a map T and $F = f \circ T \circ f^{-1}$, then $f_*\mu$ is invariant for F.

2.10 Use the invariant measure for ρ and Proposition 8 to find invariant measures for $\overline{T}$ and T.

2.11 Show that (μ, T) is ergodic if and only if $(F = f \circ T \circ f^{-1}, f_*\mu)$ is ergodic.

2.2 Consequences of the construction (optional–geometry of numbers)

2.12 [Ser, Lemma 3.3.1] Let γ and γ' be geodesics in $\mathbb{H}$ with $\gamma_{+\infty} = \gamma'_{+\infty}$. Then the cutting sequences eventually coincide.

Definition 9. Two continued fractions $\alpha = \pm[n_1; n_2, n_3, \dots], \beta = \pm[m_1; m_2, m_3, \dots]$ have the *same tails* $\mod 2$ if there exists r, s such that

$$r + s = \begin{cases} 0 \mod 2 & \text{if } \alpha\beta > 0, \\ 1 \mod 2 & \text{if } \alpha\beta < 0 \end{cases}$$

and $n_{r+k} = m_{s+k},\ k \geq 0$.

2.13 [Ser, 3.3.3] For $\alpha, \beta \in \mathbb{R}$, there exists $M \in \mathrm{PSL}(2,\mathbb{Z})$ such that $\alpha = M\beta$ if and only if they have the same tail mod 2.

2.14 [Ser, 3.3.4] A number $\alpha > 1$ has a purely periodic regular continued fraction expansion if and on if $\alpha, \overline{\alpha}$ are roots of the same quadratic polynomial with $-1 < \overline{\alpha} < 0$ and

$$\alpha = \overline{[n_1; n_2, n_3, \dots, n_{2r}]}, \quad \bar{\alpha} = -\overline{[0; n_{2r}, n_{2r-1}, \dots, n_1]}. \tag{1}$$

2.15 [Ser, 3.3.5] The regular continued fraction expansion of α is eventually periodic if and only if α is the root of a quadratic polynomial.

2.3 More continued fractions and Farey fractions (optional–no geometry)

2.16 Let $x = [0; a_1, a_2, \dots], T_n = [0; a_{n+1}, a_{n+2}, \dots]$, and $V_n = \frac{q_{n-1}}{q_n}$ for $n \geq 1$ and $V_0 = 0$. $V_n = [0; a_n, a_{n-1}, \dots, 1]$ by Lesson 2, Problem **2.3**. Show that $T^n(x) = T_n$.

2.17 Show that $\overline{T}^n(x, 0) = (T_n, V_n)$.

Definition 10. Construct a table using the following rules: In the first row, write $\frac{0}{1}$ and $\frac{1}{1}$.

For the n^{th} row, copy the $(n-1)^{st}$ row. For each $\frac{a}{b}$ and $\frac{c}{d}$ in the$(n-1)^{st}$ row, insert $\frac{a+c}{b+d}$ between $\frac{a}{b}$ and $\frac{c}{d}$ if $b + d \leq n$. The first four rows are

f_1	$\frac{0}{1}$						$\frac{1}{1}$
f_2	$\frac{0}{1}$			$\frac{1}{2}$			$\frac{1}{1}$
f_3	$\frac{0}{1}$		$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$		$\frac{1}{1}$
f_4	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

We call the sequence in the n^{th} row of the table f_n.

2.18 If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in f_n with $\frac{c}{d}$ to the left of $\frac{a}{b}$, then $ad - bc = 1$.

2.19 Every $\frac{p}{a}$ in the table is in reduced form, ie, $(p, q) = 1$. The fractions in each row are ordered from smallest to largest.

2.20 If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in any row, then for all f_n that contain $\frac{a+c}{b+d}$, $\frac{a+c}{b+d}$ has the smallest denominator of any rational number between $\frac{a}{b}$ and $\frac{c}{d}$ and is the unique rational number between $\frac{a}{b}$ and $\frac{c}{d}$ with denominator $b + d$.

2.21 If $0 \leq x \leq y, (x, y) = 1$, then the fraction $\frac{x}{y}$ appears in the y^{th} row of the table and all future rows.

2.22 The n^{th} row consists of all reduce fractions with $\frac{a}{b}$ such that $0 \leq \frac{a}{b} \leq 1$ and $0 < b \leq n$. The fractions are listed from smallest to largest. That is, f_n is the portion of the n^{th} Farey sequence between 0 and 1.

Lesson 5: Ergodicity, the pointwise ergodic theorem and mixing

In this section we describe ergodicity, the pointwise ergodic theorem and mixing.

1 Material

1.1 Ergodicity

Fix $T : [0,1) \to [0,1)$ with an absolutely continuous invariant measure ν. We would like to know if every (measurable) set $A \subset [0,1)$ has associated with it a set $N_A \subset [0,1)$ of measure zero such that

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in A)\}|}{N} = \nu(A) \tag{1}$$

holds for all $x \in [0,1) \setminus N_A$. We already saw in Lesson 3 that in general N_A cannot be empty, or even countable.

If the above were true then in particular if $\nu(A) > 0$ then the orbit of almost every point $x \in [0,1)$ must enter A at some point. (In fact, infinitely often.) Therefore, a necessary condition for the above to be true is that *every* set $A \subset [0,1)$ of positive measure sees the orbit of almost every $x \in [0,1)$. Put another way, if the above is to hold then there cannot be a set $B \subset [0,1)$ with $0 < \nu(B) < 1$ such that $T(x) \in B$ whenever $x \in B$; if that were the case the orbit of every point in B would never visit the positive measure set $[0,1) \setminus B$.

Slightly more formally, suppose that $X = A \cup B$ is a disjoint union of set A, B with $\nu(A) > 0$ and $\nu(B) > 0$ such that

- $T(x) \in A$ for all $x \in A$
- $T(x) \in B$ for all $b \in B$

both hold. It is then impossible for there to be a set N_B of zero measure such that

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in B\}|}{N} = \nu(B)$$

holds for all $x \in [0,1) \setminus N$ because $\nu(A)$ and $\nu(B)$ were assumed positive but the left hand side is zero for every point in A.

Say that $A \subset [0,1)$ is **invariant** if $A = T^{-1}A$. A map $T : [0,1) \to [0,1)$ is **ergodic** with respect to an invariant absolutely continuous measure ν if $\nu(A) \in \{0,1\}$ whenever A is a T invariant set. Informally, ergodicity is the statement that there are no invariant sets of positive measure.

Ergodicity is a notion of indecomposability, analogous to irreducibility in representation theory. If T is ergodic with respect to ν then we cannot decompose $[0,1)$ into smaller pieces in a way that is meaningful to ergodic theory and study T on each of the pieces. That is not to say it is impossible to decompose $[0,1)$ into T invariant pieces; but it does say that any such decomposition will be made of pieces of measure zero and the complements of such pieces. From the point of view of ergodic theory, there is no difference between the study of the ternary map $T(x) = 3x \bmod 1$ on $[0,1)$ and its study on $[0,1) \setminus C$ where C is the middle thirds Cantor set: our absolutely continuous invariant measure cannot detect whether you have retained C or discarded it.

1.2 The pointwise ergodic theorem

Theorem 11. *Fix $T : [0,1) \to [0,1)$ with an absolutely continuous invariant measure ν and suppose that T is ergodic for ν. For every measurable $A \subset [0,1)$ one can find $N_A \subset [0,1)$ with of zero measure such that*

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in A\}|}{N} = \nu(A)$$

for every $x \in [0,1) \setminus N_A$.

One can think of the ergodic theorem as saying that the orbit of a "random" point $x \in [0,1)$ under an ergodic transformation will visit a given set $A \subset [0,1)$ with the "correct" frequency $\nu(A)$. By "random" we mean "with a zero measure set of exceptions" and by "correct" we mean the frequency assigned to the interval by our absolutely continuous measure.

We will see many application of the ergodic theorem later. For now lets consider again the base three map $T(x) = 3x \bmod 1$ and put $A = [\frac{1}{3}, \frac{2}{3})$. The theorem tells us there is a set $N_A \subset [0,1)$ of zero measure such that

$$\lim_{N \to \infty} \frac{|\{1 \le n \le N : d_n(x) = 1\}|}{N} = \frac{1}{3} \tag{2}$$

and we conclude that, apart from a measure zero set of exceptions, a third of the digits in every ternary expansion are equal to 1.

As an aside, one might argue at this point that numbers with an asymptotic frequency of $\frac{1}{3}$ for ternary digits are not special. Why should the collection of numbers satisfying (2) be large and the collection of numbers satisfying

$$\lim_{N \to \infty} \frac{|\{1 \le n \le N : d_n(x) = 1\}|}{N} = \alpha$$

for *any* other $0 < \alpha < 1$ be of measure zero? The answer is that we have chosen a specific means ν of assigning sizes to sub-intervals of $[0,1)$ and the one we have chosen happens only to be interested in numbers satisfying (2). This is analogous to the following situation: were one to toss a bias coin, for which the probability of heads was known to be $\frac{1}{5}$, one would not be surprised to find that the collection of infinitely repeated tosses in which one does *not* find the proportion of heads to be $\frac{1}{5}$ is negligible.

The ergodic theorem is only applicable if we have an absolutely continuous invariant measure with respect to which T is ergodic. All the following transformations are known to be ergodic.

- The rotation $T(x) = x + \alpha \bmod 1$ for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
- The base b digit map $T(x) = bx \bmod 1$ for every $2 \le b \in \mathbb{N}$.
- The continued fraction map $T(x) = \frac{1}{x} \bmod 1$.

It is usually not easy to prove that a map T together with an invariant measure ν is ergodic. We will see some techniques for doing so in the problems. For now we focus on a much stronger property – mixing – that guarantees ergodicity.

1.3 Mixing

There is no randomness inherent in a dynamical system. We have a specific rule T that describes at every iteration how the points of $[0,1)$ will move around. Nevertheless, it is sometimes the case that a dynamical system can behave like a random process.

Fix $T : [0,1) \to [0,1)$ with an invariant measure ν. We say that T is **mixing** with respect to ν if

$$\lim_{n \to \infty} \nu(A \cap (T^n)^{-1} B) = \nu(A)\nu(B) \tag{3}$$

for all measurable $A, B \subset [0,1)$.

If T is mixing with respect to ν then, for any intervals A, B and any $\epsilon > 0$ one can find N so large that

$$|\nu(A \cap (T^n)^{-1} B) - \nu(A)\nu(B)| < \epsilon \nu(B)$$

for all $n \ge N$. In particular

$$\left| \frac{\nu(A \cap (T^n)^{-1} B)}{\nu(A)} - \nu(B) \right| < \epsilon$$

for all $n \ge N$. We can interpret this as a statement of independence: one can hardly do better than guess $\nu(B)$ when presented with the question "what is the probability that a point $x \in A$ will belong to B after n iterates?". In probability theory language the "events" A and B are **independent** if $\nu(A \cap B) = \nu(A)\nu(B)$. Mixing is therefore "asymptotic independence" in the sense that A and $(T^n)^{-1} B$ become closer and closer to independent as n grows.

Checking (3) holds for any pair of intervals $A, B \subset [0,1)$ could be tedius, perhaps involving an approximation of an arbitrary interval by one with, say, dyadic endpoints. Fortunately, it is not necessary to carry out such an argument repeatedly. Call a set $\mathcal{I}$ of sub-intervals of $[0,1)$ a **generating family** if any open set can be written as a countable union of intervals from $\mathcal{I}$.

For any $1 \neq b \in \mathbb{N}$ the collection

$$\left\{ \left[\frac{i}{b^k}, \frac{i+1}{b^k} \right) : k \in \mathbb{N}, 0 \le i < n^k \right\}$$

is a generating family, as is the collection of all continued fraction intervals

$$\{x \in [0,1) : [a_1, \ldots, a_{n-1}, a_n] < x < [a_1, \ldots, a_{n-1}, a_n + 1]$$

Theorem 12. *To check that T is mixing for a measure ν it is enough to check it is mixing on a generating family.*

Mixing is a very strong property for a dynamical system to have. It is strong enough to easily imply ergodicity.

Theorem 13. *If T on $[0,1)$ is mixing with respect to an absolutely continuous invariant measure ν then T is ergodic.*

Proof. Fix $B \subset [0,1)$ a T invariant set. Then $(T^n)^{-1}B = B$ for all n in $\mathbb{N}$. For every $A \subset X$ the sequence $n \mapsto \nu(A \cap (T^n)^{-1}B)$ both converges to $\nu(A)\nu(B)$ because T is mixing, and is constant. Thus $\nu(A \cap B) = \nu(A)\nu(B)$. Taking $A = B$ gives $\nu(A) = \nu(A)^2$ whence $\nu(A) = 0$ or $\nu(A) = 1$. □

2 Problems

2.1 Invariant sets and ergodicity

Recall that $T^{-1}A = \{x \in X : T(x) \in A\}$.

2.1 Explain why $T^{-1}A \subset A$ can be read as "nobody can enter A" and why $T^{-1}A \supset A$ can be read "nobody can leave A".

2.2 For each of the conditions below defining a set $A \subset [0,1)$, which ones define a set satisfying $T^{-1}A = A$?

- the ternary expansion of $x \in [0,1)$ contains the string 212 infinitely often.
- The ternary expansion of $x \in [0,1)$ contains the string 02 finitely many times.
- The ternary expansion of $x \in [0,1)$ contains the string 01 at least seven times.

2.3 Say that $x, y \in X$ are **cofinal** if there are $n, m \in \mathbb{N}$ with $T^n x = T^m y$.

(a) Prove that x and y are cofinal if and only if there is $n \in \mathbb{N}$ with $T^n(y)$ belonging to the orbit of x.

(b) Prove that cofinal is an equivalence relation and that all equivalence classes are invariant.

(c) Does the decomposition of $[0,1)$ into cofinal equivalence classes preclude any $T : [0,1) \to [0,1)$ from being ergodic?

(d) Let Ω be a subset of $[0,1)$ containing exactly one element from every equivalence class. Compute $\nu_x(\Omega)$ for every $x \in [0,1)$.

2.4 Put $T(x) = 3x \bmod 1$ and $A = [\frac{1}{3}, \frac{2}{3})$.

(a) Fix $0 \le \alpha \le 1$. Prove that the set F_α of $x \in [0,1)$ for which $\nu_x(A) = \alpha$ is T invariant.

(b) Prove that $F_\alpha \cap F_\beta = \emptyset$ whenever $\alpha \neq \beta$.

(c) Do the sets F_α for $0 \le \alpha \le 1$ cover $[0,1)$?

2.5 Let $T : [0,1) \to [0,1)$ be the map $T(x) = x + \frac{1}{3} \bmod 1$.

(a) Verify that $\phi = 1$ is an invariant density for T.

(b) Find a decomposition $[0,1) = A \cup B$ into invariant sets of positive density.

2.2 Irrational rotations

Fix $\alpha \in \mathbb{R}$ irrational and let $T : [0,1) \to [0,1)$ be the map $T(x) = x + \alpha \bmod 1$.

2.6 (a) Prove for every $N \in \mathbb{N}$ that $\{T^n(0) : 1 \leq n \leq N\}$ has cardinality N.

(b) Prove for every $N \in \mathbb{N}$ that there are $1 \leq i < j \leq N$ with $|T^i(0) - T^j(0)| < \frac{1}{N-1}$.

(c) With i, j as above verify that either $|T^{j-i}(0) - 0| < \frac{1}{N-1}$ or $|T^{j-i}(0) - 1| < \frac{1}{N-1}$ holds.

(d) Prove that the orbit $\{T^n(0) : n \in N\}$ of 0 is dense in $[0,1)$.

(e) Prove that the orbit of every point $x \in [0,1)$ is dense in $[0,1)$. (How can the orbit of x be obtained from the orbit of 0?)

2.7 Recall that $\phi = 1$ is an invariant density for T. Let ν be the corresponding measure. Fix a set $A \subset [0,1)$ that is T invariant and of positive measure. Take for granted the following consequence of the Lebesgue density theorem: there is a point $a \in A$ such that

$$\lim_{r \to 0} \frac{\nu(A \cap [a-r, a+r])}{2r} = 1$$

holds.

(a) Prove for every $\epsilon > 0$ that $\nu(A) > 1 - \epsilon$.

(b) Deduce that T is ergodic.

2.8 Fix $p \in \mathbb{Z}$ and let $\chi_p(x) = e^{2\pi i p x}$.

(a) Compute that limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_p(T^n x) = \int \chi_p(x) \, \mathsf{d}x$$

for all $p \in \mathbb{Z}$ and all $x \in [0,1)$.

(b) Use the Stone-Weierstrass theorem to determine the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

for all $x \in [0,1)$ and all continuous functions $f : [0,1) \to \mathbb{C}$.

(c) Fix $[a,b) \subset [0,1)$. By choosing appropriate continuous function $f, g : [0,1) \to [0,2)$ that satisfy $f \leq 1_{[a,b)} \leq g$ prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1_{[a,b)}(T^n x) = \nu([a,b))$$

for all $x \in [0,1)$.

(d) Conclude that the conclusion of the pointwise ergodic theorem holds for all intervals $[a,b) \subset [0,1)$.

2.3 The pointwise ergodic theorem

2.9 Fix $T : [0,1) \to [0,1)$ with an absolutely continuous invariant measure ν. Suppose that the conclusion of the pointwise ergodic theorem is known to hold for T. Prove that T is ergodic with respect to ν.

2.10 Fix $T : [0,1) \to [0,1)$ ergodic for an invariant absolutely continuous measure ν.

(a) Deduce from the ergodic theorem that there is a set N of measure zero such that every $x \in [0,1) \backslash N$ satisfies

$$\lim_{N \to \infty} \frac{|\{1 \leq n \leq N : T^n(x) \in [a,b)|}{N} = \nu([a,b)) \tag{1}$$

for all rationals $0 \leq a < b \leq 1$.

(b) Can the restriction to rational a, b be lifted?

2.11 For the map $T(x) = 3x \bmod 1$ and its invariant density $\phi = 1$ does the conclusion of the pointwise ergodic theorem hold for every point $x \in [0, 1)$?

2.12 Fix $0 \le \alpha < 1$ irrational and let $T : [0, 1) \to [0, 1)$ be the map $T(x) = x + \alpha \bmod 1$.

(a) Prove that if $N \subset [0, 1)$ has zero measure then its complement is dense.

(b) Fix an interval $[a, b) \subset [0, 1)$. Prove for T that (1) holds for every point $x \in [0, 1)$.

2.4 Mixing

2.13 Let T be the ternary map and let $I, J \subset [0, 1)$ be intervals both of the form $[\frac{i}{3^k}, \frac{i+1}{3^k})$ for some $k \in \mathbb{N}$ and some $0 \le i < 3^k$.

(a) Compute the limit $\lim_{n\to\infty} \nu(I \cap (T^n)^{-1}J)$.

(b) Does your answer imply that T is mixing?

(c) Does your answer imply that T is ergodic?

2.14 Let $T : [0, 1) \to [0, 1)$ be the Gauss map and let ν be its absolutely continuous invariant measure. Given $a_1, \ldots, a_n \in \mathbb{N}$ let $\Delta(a_1, \ldots, a_n)$ be the set of $0 \le x \le 1$ whose continued fraction expansion begins $[a_1, \ldots, a_n, \ldots]$.

(a) Compute $\nu(\Delta(a_1))$.

(b) Compute $\nu(\Delta(a_1) \cap (T^1)^{-1}\Delta(a_2))$.

(c) Would you enjoy calculating $\nu(\Delta(a_1) \cap (T^n)^{-1}\Delta(a_2))$ for all $n \in \mathbb{N}$?

Lesson 6: Computational Experiments with Mathematica

The material and problems for Lesson 6 are available in a Mathematica notebook on the website.

Lesson 7: Ergodic properties of the Gauss map

1 Material

The goal of this lesson is to explore the ergodic properties of the Gauss map introduced in Lesson 2. We start by looking at some properties of the rational approximations of a real number using continued fractions.

1.1 Another matrix representation of continued fractions

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{Z}, ad-bc = \pm 1\}$. We define the **Möbius transformation** as a map $A : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ by

$$Ax := \frac{ax+b}{cx+d}. \tag{1}$$

Definition 14. Let $x = [a_0; a_1, a_2, \dots]$. For $n \geq 1$, define A_n as

$$\begin{aligned} A_0 := \left(\begin{smallmatrix} 1 & a_0 \\ 0 & 1 \end{smallmatrix}\right), A_n &:= \left(\begin{smallmatrix} 0 & 1 \\ 1 & a_n \end{smallmatrix}\right) \\ M_n &:= A_0 A_1 \cdots A_n. \end{aligned} \tag{2}$$

We also define the **convergents** $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$.

Note that this is a slightly nonstandard matrix representation of continued fractions, where A_0 has a different form than A_n. However, this allows us to have $A_0 = I$ when $x \in [0,1)$.

Proposition 15. *Let $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$ (see that this matches with p_0, q_0 above). Then*

$$p_n = a_n p_{n-1} + p_{n-2} \tag{3}$$
$$q_n = a_n q_{n-1} + q_{n-2} \tag{4}$$
$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n \tag{5}$$

for $n \geq 1$. Considering M_n and A_n as Möbius transformations. Then

$$M_n 0 = [a_0; a_1, \dots, a_n] = \frac{p_n}{q_n}, \quad M_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}.$$

Proof. First we note that if $M_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$, then $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ and $M_n 0 = \frac{p_{n-1}0 + p_n}{q_{n-1}0+q_n} = \frac{p_n}{q_n}$. Also, $M_{n+1} = M_n A_{n+1} = \begin{pmatrix} p_n & p_{n-1} + a_{n+1}p_n \\ q_n & q_{n-1} + a_{n+1}q_n \end{pmatrix}$. Now we proceed by induction.

We can start with $n = 0$, where $[a_0] = \frac{p_0}{q_0} = \frac{a_0}{1}$, and $M_0 = A_0 = \begin{pmatrix} 1 & a_0 \\ 0 & 1 \end{pmatrix}$.

We also want to check $n = 1$, since the pattern changes after $n = 0$ (and we ignored the first column and equivalence relations in the $n = 0$ case). Now, $[a_0; a_1] = \frac{a_0 a_1 + 1}{a_1} = \frac{p_0 a_1 + p_{-1}}{q_0 a_1 + q_{-1}} = \frac{p_1}{q_1}$, and $M_1 = A_0 A_1 = \begin{pmatrix} a_0 & a_0 a_1 + 1 \\ 1 & a_1 \end{pmatrix}$.

Now assume $M_k = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix}$, $p_k = p_{k-1} + a_k p_{k-2}$ and $q_k = q_{k-1} + a_k q_{k-2}$ for $k \leq n$. Then $M_{n+1} = M_n A_{n+1} = \begin{pmatrix} p_n & p_{n-1} + a_{n+1}p_n \\ q_n & q_{n-1} + a_{n+1}q_n \end{pmatrix}$. Now we need to show that $p_{n+1} = p_{n-1} + a_{n+1}p_n$ and $q_{n+1} = q_{n-1} + a_{n+1}q_n$. By the induction hypothesis,

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n] = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

By definition,

$$\begin{aligned}\frac{p_{n+1}}{q_{n+1}} &= [a_0; a_1, a_2, \ldots, a_n, a_{n+1}] = [a_0; a_1, a_2, \ldots, a_n + \frac{1}{a_{n+1}}] \\ &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \text{ (by induction hypothesis)} \\ &= \frac{a_{n+1}\left(a_n p_{n-1} + p_{n-2}\right) + p_{n-1}}{a_{n+1}\left(a_n q_{n-1} + q_{n-2}\right) + q_{n-1}} \\ &= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} \text{ (by induction hypothesis).}\square\end{aligned}$$

1.2 The Gauss map is ergodic

We saw in Lesson 3 Problem **2.8** that $\frac{1}{\log 2}\frac{1}{1+x}$ is an invariant density for the Gauss map. We can also see that $\lambda(x) = 1$ is not an invariant density, since

$$T^{-1}\left(0, \frac{1}{2}\right) = \bigcup_{n=1} \infty \left(\frac{1}{n+\frac{1}{2}}, \frac{1}{n}\right)$$

and

$$\int\limits_{T^{-1}(0,\frac{1}{2})} 1 \, \mathsf{d}x = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}}\right) = \sum_{n=1}^{\infty} \frac{\frac{1}{2}}{n(n+\frac{1}{2})} = \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - \log 4 \neq \frac{1}{2} = \int\limits_{(0,\frac{1}{2})} 1 \, \mathsf{d}x.$$

We call $\lambda(A) = \int_A 1 \, \mathsf{d}x$ the **Lebesgue measure**. If A is an interval (a, b) (or $[a, b), (a, b], [a, b]$), then $\lambda(A) = b - a$.

We are going to use Knopp's Lemma to prove that the Gauss map is ergodic.

Lemma 16 (Knopp's Lemma). *Fix $A \subset [0, 1)$. If there is $\gamma > 0$ such that for every $I \subset [0, 1)$ one has $\nu(A \cap I) \geq \gamma\nu(I)$ then $\nu(A) = 1$.*

Theorem 17. *The Gauss map $T : [0, 1) \to [0, 1)$*

$$Tx := \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ 0 & \text{if } x = 0, \end{cases} \tag{6}$$

is ergodic with respect to the Gauss measure $\mu(A)$ with density $\frac{1}{\log 2}\frac{1}{1+x}$. That is, for all $A \subseteq [0, 1)$[2] *we have that*

$$\mu(A) = \frac{1}{\log 2} \int\limits_A \frac{1}{1+x} \, \mathsf{d}x = \frac{1}{\log 2} \int\limits_{T^{-n}A} \frac{1}{1+x} \, \mathsf{d}x = \mu(T^{-n}A)$$

if and only if $\mu(A) \in \{0, 1\}$.

Proof. We saw in Lesson 3 Problem **2.8** that $\frac{1}{\log 2}\frac{1}{1+x}$ is an invariant density for T. In Lesson 2 Problem **2.4**b we saw that $x = M_n(T^n x) = \left(\begin{smallmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{smallmatrix}\right) T^n x = \frac{p_{n-1}T^n x + p_n}{q_{n-1}T^n x + q_n}$. In Problem **2.10**, we show that $\{x : 0 \leq a \leq T^n x \leq b \leq 1\} \cap \Delta_n$ is

$$\left[\frac{p_{n-1}a + p_n}{q_{n-1}a + q_n}, \frac{p_{n-1}b + p_n}{q_{n-1}b + q_n}\right)$$

when n is even, and

$$\left(\frac{p_{n-1}b + p_n}{q_{n-1}b + q_n}, \frac{p_{n-1}a + p_n}{q_{n-1}a + q_n}\right]$$

[2]where it makes sense to define an integral

when n is odd.

Note that $T^{-n}(a,b) = \{a \le T^n x \le b\}$. In problem **2.11**, we show that the Lebesgue measure of $T^{-n}[a,b) \cap \Delta_n$ is

$$\frac{\lambda([a,b))\lambda(\Delta_n)q_n(q_{n-1}+q_n)}{(q_{n-1}b+q_n)(q_{n-1}a+q_n)}.$$

We now go through a series of estimates comparing the Lebesgue measure and the Gauss measure of $T^{-n}(a,b) \cap \Delta_n$. Many of the calculations are left for the exercises. First, we have that

$$\frac{1}{2} < \frac{q_n}{q_{n-1}+q_n} < \frac{q_n(q_{n-1}+q_n)}{(q_{n-1}b+q_n)(q_{n-1}a+q_n)} < \frac{q_n(q_{n-1}+q_n)}{q_n^2} < 2$$

Thus, for any interval $I \subseteq [0,1)$,

$$\frac{\lambda(I)\lambda(\Delta_n)}{2} < \lambda(T^{-n}I \cap \Delta_n) < 2\lambda(I)\lambda(\Delta_n).$$

Let A be a finite disjoint union of intervals I. Since the Lebesgue measure is additive (since it is an integral), we also have that

$$\frac{\lambda(A)\lambda(\Delta_n)}{2} < \lambda(T^{-n}A \cap \Delta_n) < 2\lambda(A)\lambda(\Delta_n). \tag{7}$$

This week, we are only considering measures were it is sufficient to check sets that are a collection of finite disjoint unions of intervals. In problem **2.12**, we see that

$$\frac{\lambda(A)}{2\log 2} \le \mu(A) \le \frac{\lambda(A)}{\log 2}. \tag{8}$$

Combining (7) and (8), we find that

$$\mu(T^{-n}A \cap \Delta_n) \ge \frac{\log 2\mu(A)\mu(\Delta_n)}{4}.$$

Now, suppose that B is an invariant set with $\mu(B) > 0$. Since Δ_n is a fundamental interval, we can use Knopp's lemma with $\gamma = \frac{\log 2}{4}\mu(B)$. Thus, $\mu(B) = 1$ and T is ergodic with respect to μ. $\square$

2 Problems

2.1 Matrix representation of continued fractions

2.1 ([DK] Exercise 1.3.6) Show that if $A, B \in \mathrm{GL}(2,\mathbb{Z})$, then $(AB)x = A(Bx)$.

2.2 ([DK] Exercise 1.3.8) Show:

(a) $(p_n, q_n) = 1$,

(b) $p_n(x) = q_{n-1}(Tx)$ for all $n \ge 1$,

(c) the sequence $(q_n)_{n\ge1}$ is a monotone increasing sequence of positive integers.

2.3 ([DK] Exercise 1.3.9) Use the recurrence relation for q_n to show

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_1].$$

2.4 ([DK] Exercise 1.3.10) This exercise will prove that $\lim_{n\to\infty} \frac{p_n}{q_n} = x$. We define one more matrix

$$A_n^* := \begin{pmatrix} 0 & 1 \\ 1 & a_n + T_n \end{pmatrix}, n \ge 1, T_n = T^n x.$$

(a) Show that $x = (M_{n-1}A_n^*)0$.

(b) Use the the fact that

$$M_{n-1} = \begin{pmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{pmatrix}, n \geq 1$$

and the recurrence relations for $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ to show that

$$x = \frac{p_n + T_n p_{n-1}}{q_n + T_n q_{n-1}}, \quad n \geq 1$$

ie, $x = M_n(T^n x)$.

(c) Use the fact that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ to conclude that

$$x - \frac{p_n}{q_n} = \frac{(-1)^n T_n}{q_n(q_n + T_n q_{n-1})}, \quad n \geq 1.$$

(d) Since $T_n \in [0, 1)$ we have that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}, \quad n \geq 1.$$

(e) $\lim_{n\to\infty} \frac{p_n}{q_n} = x$.

2.5 Here is an alternate proof of the fact that $\lim_{n\to\infty} \frac{p_n}{q_n} = x$. Use the fact that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$ to conclude that

$$\frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

and thus $\lim_{n\to\infty} \frac{p_n}{q_n} = x$.

2.6 ([DK] Exercise 1.3.11) Show that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < x < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Define **cylinder set**

$$\Delta_n = \Delta(a_1, a_2, \ldots, a_n) := \{x \in (0, 1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

The cylinder sets are fundamental intervals of rank n for the Gauss map.

2.7 ([DK] Exercise 1.3.13) Show that $\Delta(1) = \left(\frac{1}{2}, 1\right)$ and $\Delta(n) = \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \geq 2$. Determine $\Delta(1, 1)$ and $\Delta(m, n)$ for $m, n \geq 1$.

2.8 ([DK] Exercise 1.3.15) Show that $\Delta(a_1, a_2, \ldots, a_k)$ is an interval in $[0, 1)$ with endpoints

$$\frac{p_k}{q_k} \quad \text{and} \quad \frac{p_k + p_{k-1}}{q_k + q_{k-1}}.$$

2.2 Details of ergodicity proof

2.9 Find $\lambda(\Delta(a_1, \ldots, a_n))$ and $\mu(\Delta_n(a_1, \ldots, a_n))$.

2.10 Check that $\{x : 0 \leq a \leq T^n x \leq b \leq 1\} \cap \Delta_n$ is

$$\left[\frac{p_{n-1}a + p_n}{q_{n-1}a + q_n}, \frac{p_{n-1}b + p_n}{q_{n-1}b + q_n}\right)$$

when n is even, and

$$\left(\frac{p_{n-1}b + p_n}{q_{n_1}b + q_n}, \frac{p_{n-1}a + p_n}{q_{n-1}a + q_n}\right]$$

when n is odd.

2.11 ([DK] Exercise 3.5.2) Show that the Lebesgue measure of $T^{-n}[a,b) \cap \Delta_n$ is

$$\frac{\lambda([a,b))\lambda(\Delta_n)q_n(q_{n-1}+q_n)}{(q_{n-1}b+q_n)(q_{n-1}a+q_n)}.$$

2.12 Verify that $\frac{\lambda(I)}{2\log 2} \leq \mu(1) \leq \frac{\lambda(I)}{\log 2}$ for any interval $I \subseteq [0,1)$. Explain why these inequalities hold if we replace I with A which is a finite disjoint union of intervals.

2.3 Application of the Ergodic Theorem to continued fractions

2.13 [Lév] Let $a \in \mathbb{Z}_+$ be given and let $x \in [0,1)$, with continued fraction expansion $x = [a_1, a_2, \dots]$. Then for almost all $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}\#\{1 \leq i \leq n; a_i = a\} = \frac{1}{\log 2}\log\left(1 + \frac{1}{a(a+2)}\right).$$

***2.14** [Lév] $\lim_{n\to\infty} \frac{1}{n}\log q_n = \frac{\pi^2}{12\log 2}$

2.15 $\lim_{n\to\infty} \frac{1}{n}\log(\lambda(\Delta_n) = \frac{-\pi^2}{6\log 2}$

2.16 [Lév] $\lim_{n\to\infty} \frac{1}{n}\log\left|x - \frac{p_n}{q_n}\right| = \frac{-\pi^2}{6\log 2}$

2.17 [Khi1] For almost every $x \in [0,1)$ with continued fraction expansion $x = [a_1, a_2, \dots]$,

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = 1.7454065\dots$$

2.18 [Khi1] For almost every $x \in [0,1)$ with continued fraction expansion $x = [a_1, a_2, \dots]$,

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = \infty.$$

***2.19** [Khi1] For almost all x

$$\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{1/n} = \prod_{k=1}^{\infty}\left(1 + \frac{1}{k(k+1)}\right)^{\frac{\log k}{\log 2}} = 2.6854\dots$$

Lesson 8: Basics of Diophantine approximation

1 Material

The subject of **Diophantine Approximation** concerns approximating real numbers with rational numbers. It is not immediately clear which questions lead to interesting answers.

(I) How close can we get to $x \in \mathbb{R}$ with a rational number p/q? A boring answer: if $x \in \mathbb{Q}$, then we can find $p/q = x$, and if $x \notin \mathbb{Q}$, then we can find p/q as close to x as we please since the rational numbers are dense in the real numbers.

(II) How close can we get to $x \in \mathbb{R}$ with a rational number with denominator equal to q? A not so interesting answer: the set of rational numbers that can be written with a fixed denominator of q is $1/q$-dense in the real numbers, so there is always a rational number p/q satisfying $|x - p/q| \leq 1/(2q)$.

Tip: If you've only got boring answers to your questions, you're asking the wrong questions.

(III) How close can we get to $x \in \mathbb{R}$ with a rational number with denominator less than or equal to $Q \in \mathbb{N}$? This question is much more complex and leads us to the fertile ground of Diophantine Approximation.

From now on, whenever we write p/q, it is tacitly assumed that $q > 0$ and p and q are coprime so that p/q is in lowest terms. The **height** of the non-zero rational p/q is defined to be q.

We will also need the following notation: for a real number x, $\{x\}$ denotes its fractional part and $\|x\|$ denotes the distance to the nearest integer.

1.1 Farey fractions and Dirichlet's theorem

Listing all the rational numbers in the interval $[0, 1]$ of height at most Q, we get $\mathcal{F}_Q$, the Q^{th} **Farey sequence**. The first few Farey sequences are:

$$\begin{aligned}
\mathcal{F}_1 &= \left\{\frac{0}{1}, \frac{1}{1}\right\}, \\
\mathcal{F}_2 &= \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \\
\mathcal{F}_3 &= \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}, \\
\mathcal{F}_4 &= \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}.
\end{aligned}$$

Question (III) above asks, in some sense, about how evenly distributed the set $\mathcal{F}_Q$ is in the interval $[0, 1]$.

One way to measure the distribution of a set $A \subseteq [0, 1]$ is to list the elements in order $A = \{a_0 < \ldots < a_k\}$ and compute the quantity

$$D(A) = \sum_{i=0}^{k} \left| a_i - \frac{i}{k} \right|.$$

The number $D(\mathcal{F}_Q)$ quantifies how far $\mathcal{F}_Q$ is from being as evenly distributed as possible. Since we believe that $\mathcal{F}_Q$ is pretty evenly distributed, we could hope to show, for example, that for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that $D(\mathcal{F}_Q) < C_\epsilon Q^{1/2+\epsilon}$. In fact, this statement can be shown to be equivalent to the Riemann Hypothesis [BvdPSZ, Page 22]!

Another way to measure the distribution of a set $A \subseteq [0, 1]$ is to compute the quantity

$$\Psi(A) = \sup_{x \in [0,1]} d(x, A),$$

where $d(x, A) = \inf_{a \in A} |x - a|$ is the shortest distance from the point x to the set A. Another way to put this is that $\Psi(A)$ is the smallest number such that the following statement holds: for all $x \in [0, 1]$, there exists $a \in A$ such that $|x - a| \leq \Psi(A)$. Question (III) can be interpreted as asking specifically about the quantity $\Psi(\mathcal{F}_Q)$. We don't pursue this thread or the previous one any further but include them only to add context and depth around the intricacies of Question (III).

Perhaps the earliest and most famous first answer to Question (III) was given by Dirichlet in 1842. His proof, outlined in the exercises, is frequently cited as the source of the first statement of the pigeonhole principle.

Theorem 18 (Dirichlet's theorem, 1842)**.** *For all $x \in \mathbb{R}$ and $Q \in \mathbb{N}$, there exists $p/q \in \mathbb{Q}$ with $q \leq Q$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ}. \tag{1}$$

In particular, if x is irrational, there exist infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{2}$$

Dirichlet's theorem measures the degree to which a real number can be approximated by a rational number *as a function of the height of the rational number.* It makes sense to use the height, since as we allow larger and larger denominators, the degree of the approximation improves.

As an example, suppose we seek rational approximations as guaranteed by Dirichlet's theorem for the irrational number e. A natural approach would be to use the decimal approximation $e \approx 2.718281828\ldots$ to approximate e by the rational numbers $2/1$, $27/10$, $271/100$, etc.... It is true that $|e - 2/1| < 1/1^2$, but then $|e - 27/10| > 1/10^2$ and $|e - 271/100| > 1/100^2$. (Is it ever true again that the rational approximation gotten from the decimal expansion of e, after being reduced, satisfies (2)?)

Another natural approach would be to truncate the infinite series expression $e = \sum_{n=0}^{\infty} 1/n!$ to approximate e by the rational numbers $1/1$, $2/1$, $5/2$, $8/3$, $65/24$, etc.... Of these, only $2/1$, $5/2$, and $8/3$ satisfy (2). (Is it ever true again that the rational approximation of e gotten from $e = \sum_{n=0}^{\infty} 1/n!$ satisfies (2)?)

These first approaches to finding good rational approximations to e have failed. Dirichlet's theorem gives the existence of good rational approximations (for example, $19/7$ is quite good), but the method for finding such good rational approximations suggested by the proof of the theorem is quite intensive. We will see in Lesson 10 how the continued fraction convergents of e (of which $19/7$ is one) provide the best rational approximations to e.

1.2 Badly approximable and very well approximable numbers

There are a number of ways in which Dirichlet's theorem can be modified to yield new and interesting directions in Diophantine Approximation.

It is natural to ask about improving (2) by reducing the constant 1. In the exercises, you will show that for all irrational x, there are infinitely many p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

Hurwitz improved 2 to $\sqrt{5}$, which you will show in the exercises is the best possible constant: if $\sqrt{5}$ is replaced by any larger number, there are real numbers for which the statement is not true. Disregarding these obstructions, we can improve the constant $\sqrt{5}$ to $2\sqrt{2}$, and there is a new set real numbers that prevent us from improving the result further. Disregarding those allows us to improve $2\sqrt{2}$ to $\sqrt{221}/5$, $\sqrt{1517}/13$, and so on.... This this the sequence of **Lagrange numbers**, an increasing sequence of real numbers tending to 3.

The real numbers that form obstructions as described in the previous paragraph are members of the larger set of **badly approximable** numbers: irrational numbers x for which there exists $c > 0$ such that for all p/q,

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

The golden ratio $(1+\sqrt{5})/2$ is badly approximable, a fact you will establish in the exercises. In fact, every quadratic irrational is badly approximable. There are many, many more badly approximable numbers, but it is not clear why from the definition. In Lesson 10, we will use continued fractions to easily furnish uncountably many badly approximable numbers, and we will use Ergodic Theory to show that the set of badly approximable numbers has measure zero.

Besides quadratic irrationals, are any other algebraic irrationals badly approximable? Number theorists guess not, but this remains an open problem! Nevertheless, there is way that we can make sense of the statement, *all algebraic numbers are poorly approximable.* Liouville's theorem asserts that if x is algebraic of degree d, then there exists $c > 0$ such that for all p/q,

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^d}.$$

This great theorem, which you will prove in the exercises, led Liouville in 1844 to the first construction of a transcendental number. You will verify in the exercises, for example, that $\sum_{n=1}^{\infty} 10^{-n!}$ is a transcendental number. Liouville's theorem and its improvements have a long and interesting history; search for the Thue-Siegel-Roth theorem if you're curious.

Just because x is not badly approximable does not mean that it can be very well approximated. An irrational number is **very well approximable** if there exists $\epsilon > 0$ such that there are infinitely many p/q satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

You will show in the exercises that the set of very well approximable numbers has zero measure. In light of this result and the others above, the exponent of 2 in Dirichlet's theorem is the "right one" for measuring how well rational numbers approximate typical real numbers.

1.3 Shrinking targets and the Littlewood conjecture

Some questions in Diophantine Approximation can be reformulated in terms of dynamics. In this section, we will explore a connection between Dirichlet's theorem and rotations on the circle.

Fix $x \in \mathbb{R}$. Dirichlet's theorem is equivalent to the statement that for all $Q \in \mathbb{N}$, there exists $1 \leq q \leq Q$ such that

$$\|qx\| < \frac{1}{Q}.$$

The connection arises from multiplying both sides of the inequality in (1) by q. Analyzing the cases where x is rational and irrational separately, we see that for every real number x, there are infinitely many $q \in \mathbb{N}$ such that

$$\|qx\| < \frac{1}{q}.$$

This inequality is profitably interpreted in the context of rotations on a circle.

Fix $x \in \mathbb{R}$, and consider the rotation $T : [0,1) \to [0,1)$ defined by $T(y) = y + x$. By identifying 0 and 1, we understand T as a rotation on a circle of circumference 1. The orbit of the point zero is described as $T^n(0) = \{nx\}$. Note that $T^n(0)$ is within an ϵ distance of 0 on the circle if and only if $\{nx\} < \epsilon$ or $\{nx\} > 1 - \epsilon$, which happens if and only if $\|nx\| < \epsilon$. Thus Dirichlet's theorem gives that *there are infinitely many times q for which $T^q(0)$ is within a distance of $1/q$ from zero on the circle, that is, $\|qx\| < 1/q$.*

In this context, Dirichlet's theorem provides an answer to a "shrinking target problem." For $n \in \mathbb{N}$, let $I_n = [0, 1/(2n)) \cup (1 - 1/(2n), 1)$; this is an interval of length $1/n$ about 0 on the circle. These intervals are "targets" for the orbit of 0 under the irrational rotation T: we hit the target at time n if $T^n(0) \in I_n$. Note that the targets are nested and shrinking in length. The **shrinking target problem** asks whether or not the orbit of zero under the irrational rotation by $x \in \mathbb{R}$ hits the target infinitely many times. Dirichlet's theorem implies that the answer is "yes" for all $x \in \mathbb{R}$.

Some simple modifications to this shrinking target problem lead to some important problems in Diophantine Approximation. Allowing for shrinking, nested[3] targets of varying lengths leads to **Khinchin's**

[3]Khinchin's theorem requires that the sequence $q \mapsto q\Psi(q)$ is non-increasing. This implies in particular that the targets are nested.

theorem: *almost every $x \in \mathbb{R}$ is such that there are infinitely many solutions to $\|qx\| < \psi(q)$ if and only if $\sum_{q=1}^{\infty} \psi(q)$ diverges.* Lifting the nestedness assumption in Khinchin's theorem leads to the **Duffin-Schaeffer conjecture**, which was one of the most important unsolved problems in this corner of Diophantine Approximation, until a proof was announced last year (2019).

Let $x \in \mathbb{R}$. It follows from Dirichlet's theorem that

$$\liminf_{q \in \mathbb{N}} q\|qx\| \leq 1.$$

It follows immediately from the definition that x is badly approximable if and only if

$$\liminf_{q \in \mathbb{N}} q\|qx\| > 0.$$

The **Littlewood conjecture** asserts that *for all real numbers x and y,*

$$\liminf_{q \in \mathbb{N}} q\|qx\|\|qy\| = 0.$$

You will show in Lesson 10 that the set of pairs (x, y) that do not satisfy the Littlewood conjecture has zero measure. Einsiedler, Katok and Lindenstrauss showed in 2006 that the set of exceptional pairs (x, y) has zero Hausdorff dimension. To show that set of exceptional pairs is empty is one of the most famous open problems in the subject of simultaneous Diophantine Approximation!

2 Problems

2.1 Diophantine Approximation starters

2.1 Show that if $p/q \neq 5/7$, then

$$\left|\frac{5}{7} - \frac{p}{q}\right| \geq \frac{1}{7q}.$$

Formulate and prove a similar result where $5/7$ is replaced with an arbitrary rational number r/s. [4]

2.2 Use the results of the previous problem to show that if a real number x has the property that for all $\epsilon > 0$, there exists $p/q \neq x$ such that $|x - p/q| < \epsilon/q$, then x is irrational. As an application, show that

$$\sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

is irrational.

2.3 Plot the line $p = xq$ and the set of solutions to the inequality $|x - p/q| < 1/q^2$ in the qp-plane. What does Dirichlet's theorem say about lattice points in this "funnel?" How does the picture change when x is rational or irrational? Find all several lattice points (p, q) in the pq-plane such that $|\pi - p/q| < 1/q^2$.

2.4 Call p/q a **best approximation to** x if it minimizes the distance to x amongst all rational numbers with height at most q. Find the first 10 best approximations to π. Open ended question: Ask and attempt to answer at least 2 questions about best approximations.

2.5 For each rational number p/q, draw a circle in the plane of radius $1/(2q^2)$ with center $(p/q, 1/(2q^2))$. Observe that the circles corresponding to consecutive terms in a Farey sequence are tangent. Explain why.

*__2.6__ Very open ended question: What does Dirichlet's theorem say about the distribution of the Farey sequences in the interval $[0, 1]$?

[4] We may be tempted to conclude from this problem that approximating rational numbers with rational numbers is quite boring. The related inequality $|r/s - p/q| < L/q$ rearranges to $|rq - ps| < L$, a Diophantine inequality the solutions to which are important to understand in a number of situations in Number Theory.

2.2 Proof of Dirichlet's theorem

2.7 Partition $[0, 1)$ into Q-many half-open intervals of length $1/Q$. Use the pigeonhole principle to argue that there exist $0 \leq q_1 < q_2 \leq Q$ such that $\{q_1 x\}$ and $\{q_2 x\}$ belong to the same interval.

2.8 Using the definition of the fractional part function, show that there exists $q \in \{1, \ldots, Q\}$ and $p \in \mathbb{Z}$ such that $|qx - p| < 1/Q$. Finish the proof of Dirichlet's theorem in one step.

2.9 Conclude from Dirichlet's theorem that when x is irrational, there are infinitely many rational numbers p/q satisfying (2).

2.3 Approximating algebraic numbers

2.10 Let $\varphi = (1 + \sqrt{5})/2$ be the golden ratio and $\overline{\varphi} = (1 - \sqrt{5})/2$ be its conjugate. Show that

$$x^2 - x - 1 = (x - \varphi)(x - \overline{\varphi}) \qquad \text{and} \qquad \varphi = \overline{\varphi} + \sqrt{5}.$$

2.11 Let $f(x) = x^2 - x - 1$. Show that $|f(p/q)| \geq 1/q^2$.

2.12 Use the previous two exercises to show that if $|\varphi - p/q| < 1/q^2$, then

$$\left|\varphi - \frac{p}{q}\right| = \frac{|f(p/q)|}{|\overline{\varphi} - p/q|} \geq \frac{1}{\sqrt{5}q^2 + 1}.$$

Conclude that the constant $\sqrt{5}$ in Hurwitz improvement of Dirichlet's theorem cannot be improved and that the golden ratio is a badly approximable number.

2.13 Show that if $f \in \mathbb{Z}[x]$ and $f(p/q) \neq 0$, then $|f(p/q)| \geq 1/q^{\deg f}$.

2.14 A real number $x \in \mathbb{R}$ is **algebraic** if it is a root of a non-zero polynomial with rational coefficients. Following the arguments in the previous exercises, show that for every irrational algebraic number $x \in \mathbb{R}$, there exists $d \geq 2$ and $C > 0$ such that for all p/q,

$$\left|\varphi - \frac{p}{q}\right| \geq \frac{C}{q^d}.$$

In this sense, all irrational algebraic numbers are poorly approximable.

2.15 (Liouville) Show that for every $d \in \mathbb{N}$, there exist infinitely many p/q such that

$$0 < \left|\sum_{n=1}^{\infty} \frac{1}{10^{n!}} - \frac{p}{q}\right| < \frac{1}{q^d}.$$

Conclude that $\sum_{n=1}^{\infty} 10^{-n!}$ is not algebraic and hence must be **transcendental**.

2.4 The typical approximation exponent is two

This sequence of exercises leads to a complete proof of the following theorem: *The set of $x \in [0, 1]$ for which there exists $\epsilon > 0$ such that there are infinitely many solutions to the inequality*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}} \tag{1}$$

has measure zero. In this sense, Dirichlet's theorem has the best possible exponent of 2 for "most" numbers.

2.16 Suppose that for each $q \in \mathbb{N}$, the set $X_q \subseteq [0, 1]$ is a union of intervals of total length $|X_q|$. Suppose further that $\sum_{q \in \mathbb{N}} |X_q| < \infty$. Show that the set of points $x \in [0, 1]$ that belong to infinitely many of the X_q's has measure zero.

2.17 Fix $\epsilon > 0$ and $q \in \mathbb{N}$. Give an explicit description as a finite union of intervals of the set X_q of points $x \in [0, 1]$ for which there exists $p \in \mathbb{N}$ such that (1) holds.

2.18 Show that $\sum_{q\in\mathbb{N}} |X_q| < \infty$. Conclude that for a fixed $\epsilon > 0$, the set of x's for which there are infinitely many solutions to (1) has measure zero.

2.19 Complete the proof of the theorem by combining the results from the previous exercises.

Lesson 9: Applications of ergodic theory

In this section we describe some applications of ergodic theory to number theory.

1 Material

1.1 The pointwise ergodic theorem

We repeat here the pointwise ergodic theorem.

Theorem 19. *Let $T : [0,1) \to [0,1)$ have an absolutely continuous invariant measure ν. Suppose that ν is ergodic for T. Then for every piecewise continuous $f : [0,1) \to [0,\infty)$ there is a set $N_f \subset [0,1)$ of zero measure such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f \, \mathsf{d}\nu$$

for all $x \in [0,1) \setminus N_f$.

If one applies the above version of the pointwise ergodic theorem to the function $f = 1_A$ for $A \subset [0,1)$ an interval then one obtains the pointwise ergodic theorem stated in Lesson 5. Let's state the pointwise ergodic theorem at the level of generality one would usually see it.

Theorem 20. *Fix a probability space $(X, \mathscr{B}, \mu)$ and a measurable map $T : (X,\mathscr{B}) \to (X,\mathscr{B})$ suc that $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathscr{B}$. For every measurable $f : X \to \mathbb{R}$ satisfying*

$$\int |f| \, \mathsf{d}\mu < \infty$$

there is a set $N_f \in \mathscr{B}$ with $\mu(N_f) = 0$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \tag{1}$$

exists for all $x \in X \setminus N_f$. Moreover, if μ is ergodic for T then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f \, \mathsf{d}\mu$$

for all $x \in [0,1) \setminus N_f$.

In particular, the limit exists whether or not we have ergodicity. When the system is ergodic we can easily say what the limit.

1.2 Applications to continued fractions

In Lesson 7 we saw that the Gauss map is ergodic. We can apply ergodicity of the Gauss map to immediately deduce the frequency with which any $b \in \mathbb{N}$ occurs as a digit in the continued fraction expansion of almost every $x \in [0,1)$. Indeed, fixing

$$I = \left[\frac{1}{b+1}, \frac{1}{b}\right)$$

ergodicity gives us a set $N_I \subset [0,1)$ of zero measure such that

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N : a_n(x) = b\}|}{N} = \frac{1}{\log 2} \int\limits_I \frac{1}{x+1} \, \mathsf{d}x$$

holds. In particular, for almost every $x \in [0,1)$ the frequency with which the digit b occurs in the continued fraction expansion of x is

$$\int_{\frac{1}{b+1}}^{\frac{1}{b}} \frac{1}{(1+x)\log 2}\,\mathsf{d}x = \frac{1}{\log 2}\log\left(\frac{(b+1)^2}{b(b+2)}\right)$$

so that in particular 1 occurs about 41.5037% of the time whereas 8 occurs only about 1.7922% of the time. It is also possible to prove more delicate results about continued fraction digits. Each of the following results

- $\lim\limits_{n\to\infty} \frac{1}{n}\log q_n(x) = \frac{\pi^2}{12\log 2}$
- $\lim\limits_{n\to\infty} \frac{1}{n}\log \lambda(\Delta_n(x)) = -\frac{\pi^2}{6\log 2}$
- $\lim\limits_{n\to\infty} \frac{1}{n}\log \left| x - \frac{p_n(x)}{q_n(x)}\right| = \frac{-\pi^2}{6\log 2}$

is true for almost every $x \in [0,1)$ and we will see some of the details in the problem set.

A further interesting result about the digits of continued fractions is related to the geometric mean of the continued fraction digits. The **geometric mean** of positive numbers $y_1,\ldots,y_n$ is

$$\sqrt[n]{y_1\cdots y_n} = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}$$

and we always have

$$\frac{y_1+\cdots+y_n}{n} \geq \sqrt[n]{y_1\cdots y_n}$$

which is the arithmetic-geometric mean inequality.

Theorem 21. *One has*

$$\lim_{n\to\infty} \sqrt[n]{a_1\cdots a_n} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+1)}\right)^{\frac{\log k}{\log 2}} \approx 2.6854$$

for almost all $x \in [0,1)$.

We will see in the exercises that the arithmetic average

$$\lim_{N\to\infty} \frac{a_1(x)+\cdots+a_N(x)}{N} = \infty$$

for almost all $x \in [0,1)$. Despite the fact that small digits appear far more often in the continued fraction expansion of a typical real number $x \in [0,1)$ large digits appear often enough to make the average of the first N digits diverge.

1.3 Mixing of the Gauss map

We say in the Lesson 5 problem set that the ternary map $T(x) = 3x \bmod 1$ is mixing. That is to say: for any measurable sets $A, B \subset [0,1)$ we have

$$\lim_{n\to\infty} \nu(A \cap (T^n)^{-1}B) = \nu(A)\nu(B)$$

and indeed it suffices to check the above limit in the case that A, B are fundamental intervals.

Is the Gauss map mixing? It is, but it is not as easy to prove as mixing for the ternary map. We will see in the exercises that

$$\nu(\Delta_2(a_1,a_2)) \neq \nu(\Delta_1(a))\nu(\Delta_1(a_2))$$

where was we certainly have

$$\lambda(I_{ij}) = \lambda(I_i)\lambda(I_j)$$

where λ is the invariant measure for the ternary map.

There are several approaches to mixing of the Gauss map, all of which could serve as the "next step" in learning about ergodic theory.

1.1 Analyzing the transfer operator

$$(Gf)(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right)$$

on the right Banach space of functions.

1.2 Proving that the Gauss map is exact, and that exact transformations are mixing. This leads to studying entropy of dynamical systems.

1.3 Studying the invertible extension of the Gauss map.

1.4 Other applications to number theory

We mention here two other applications of ergodic theory to number theory more broadly. These are two of the hallmark contributions of ergodic theory in the past fifty years.

Furstenberg's proof of Szemerédi's theorem A set $E \subset \mathbb{N}$ has positive density if

$$\liminf_{N\to\infty} \frac{|\{1 \leq n \leq N : n \in E\}|}{N} > 0$$

and one can think of sets of positive density as taking up a positive proportion of the natural numbers.

Theorem 22 (Szemerédi)**.** *If $E \subset \mathbb{N}$ has positive density then for every $k \in \mathbb{N}$ there are $a, d \in \mathbb{N}$ such that $\{a, a+d, \ldots, a+kd\} \subset E$.*

Furstenberg gave an alternative proof of Szemerédi's theorem using ergodic theory. Frstenberg proved that the following theorem, and proved that the theorem implies Szemerédi's theorem.

Theorem 23. *For every measure-preserving transformation T of a probability space $(X, \mathscr{B}, \nu)$ and every $B \in \mathscr{B}$ and every $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that*

$$\nu(B \cap (T^n)^{-1}B \cap \cdots \cap (T^{kn})^{-1}B) > 0 \tag{2}$$

holds.

In particular, for every $T : [0,1) \to [0,1)$ with an absolutely continuous invariant measure ν and every $B \subset [0,1)$ wih $\nu(B) > 0$ the inequality (2) holds.

The new perspective on density combinatorics afforded by Furstenberg's proof led to many new results, including playing a role in the proof of the Green-Tao theorem that the set of prime numbers contain arithmetic progressions of all lengths.

Margulis's resolution of the Oppenheim conjecture A quadratic form is any map $Q : \mathbb{R}^n \to \mathbb{R}$ of the form $Q(x) = x^{\mathsf{T}}Ax$ for some $n \times n$ symmetric matrix A. For example, if

$$A = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

then

$$Q(x) = x_1(2x_1 + \sqrt{3}x_2) + x_2(\sqrt{3}x_1 + 2x_2) = 2x_1^2 + 2\sqrt{3}x_1x_2 + 2x_2^2$$

The Oppenheim conjecture is about whether the image of $\mathbb{Z}^n$ under Q is dense in $\mathbb{R}$. The **Oppenheim conjecture** is that the following three requirements together imply $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

- A is not a multiple of a matrix with rational entries.
- Q is not definite i.e. there are x, y with $Q(x) > 0$ and $Q(y) < 0$.
- $n \geq 3$.

After some thought, the first two requirements are not so surprising. If A is a multiple of a rational form then $Q(\mathbb{Z}^n)$ cannot be dense in $\mathbb{R}$.

Conjecture 24 (Oppenheim)**.** *If a quadratic form Q in at least three variables is not a multiple of a rational form and is not definite then $Q(\mathbb{R}^n)$ is dense in $\mathbb{R}$.*

This conjecture was resolved by Margulis using homogeneous dynamics; specifically by understanding the ergodic theory of how the group $\mathsf{SO}(2,1)$ acts on the coset space $\mathsf{SL}(3,\mathbb{R})/\mathsf{SL}(3,\mathbb{Z})$.

2 Problems

2.1 Digits of powers

2.1 Let f_n be the left-most digit in the decimal expression of 2^n. For example $f_5 = 3$.

(a) Verify that $f_n = k$ if and only if $\log_{10}(k) \leq \{n \log_{10}(2)\} < \log_{10}(k+1)$.

(b) Verify that $\log_{10}(2)$ is irrational.

(c) Use ergodicity of the irrational rotation $T(x) = x + \log_{10}(2)$ to compute the asymptotic frequency of $n \in \mathbb{N}$ for which f_n is 5.

2.2 Use the ideas in the previous problem to prove that every decimal expression appears as the left-most portion of the decimal representation of some power of two. (Hint: For irrational rotations the conclusion of the pointwise ergodic theorem holds for *all* points in $[0,1)$. Why? Cf. Lesson 5 Problem 2.12.)

2.2 Continued fraction applications

2.3 Take for granted that the pointwise ergodic theorem is true for piecewise continuous functions f. That is, if T is ergodic with respect to an absolutely continuous invariant measure ν and $f : [0,1) \to [0,\infty)$ is piecewise constant then there is $N_f \subset [0,1)$ of zero measure such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int f(x)\phi(x)\,\mathsf{d}x$$

holds for all $x \in [0,1) \setminus N_f$.

(a) For T the Gauss map which choice of f makes

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x)$$

the average of the digits $a_2, \ldots, a_{N+1}$ in the continued fraction expansion of x?

(b) What does the pointwise ergodic theorem tell us about the average of the continued fraction digits of almost all $x \in [0,1)$?

(c) Which choice of f will tell us about the geometric mean $\sqrt[n]{a_2 \cdots a_{n+1}}$ of the continued fraction digits $a_2, \ldots, a_{n+1}$ of almost all $x \in [0,1)$? What is the result?

2.4 In this problem we will look into the more delicate question of the limit

$$\lim_{n\to\infty} \frac{1}{n} \log q_n$$

for almost all $x \in [0,1)$ where $q_n(x)$ are the denominators of the continued fraction approximants of x. The evaluation of this limit goes back to Lévy in 1929.

(a) Use $p_n(x) = q_{n-1}(Tx)$ from the Lesson 2 exercises to write

$$-\log q_n(x) = \log \frac{p_n(x)}{q_n(x)} + \log \frac{p_{n-1}(T(x))}{q_{n-1}(T(x))} + \cdots + \log \frac{p_1(T^{n-1}(x))}{q_1(T^{n-1}(x))}$$

(b) Why is replacing $-\frac{1}{n} \log q_n(x)$ with

$$\frac{1}{n}\left(\log x + \log(Tx) + \cdots + \log(T^{n-1}x)\right)$$

not entirely unreasonable? (The estimates here are a little involved, with the details below.)

(c) Apply the pointwise ergodic theorem to $f(x) = \log x$ to deduce the limit in $\frac{\pi^2}{12 \log 2}$.

In the remaining steps we will estimate the difference

$$R(x, n) = -\log(x) - \log(Tx) - \cdots - \log(T^{n-1}x) - \log q_n(x)$$

and show that it converges to zero after dividing by n.

(d) Let $F_1, F_2, F_3, F_4, F_5, F_6, \ldots$ be the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$. Prove that $q_n(y) \geq F_n$ for all $n \in \mathbb{N}$ and all $y \in [0, 1)$.

(e) Use the mean value theorem on an appropriate interval to deduce that

$$0 < \log x - \log \frac{p_n(x)}{q_n(x)} < \frac{1}{q_n(x)}$$

when n is even. Find an analogous statement for n odd.

(f) Conclude that the remainder $R(x, n)$ is bounded by

$$\frac{1}{F_1} + \frac{1}{F_2} + \cdots + \frac{1}{F_n}$$

for all $n \in \mathbb{N}$ and all $x \in [0, 1)$.

(g) Prove that the above series converges using

$$F_n = \frac{\phi^n + (-1)^{n+1}\overline{\phi}^n}{\sqrt{5}}$$

and conclude the proof.

2.3 Szemerédi's theorem

2.5 Let $T : [0, 1) \to [0, 1)$ be the ternary map $T(x) = 3x \bmod 1$ and let ν be its absolutely continuous invariant measure. Fix $I = [\frac{i}{3^k}, \frac{i+1}{3^k})$ for some $k \in \mathbb{N}$ and some $0 \leq i < 3^k$. Prove that

$$\lim_{n\to\infty} \nu(I \cap (T^n)^{-1}I \cap \cdots \cap (T^{dn})^{-1}I) = \nu(I)^{d+1}$$

for all $d \in \mathbb{N}$.

2.6 Fix $\alpha \in \mathbb{R}$ irrational. Let $T : [0, 1) \to [0, 1)$ be the irrational rotation $T(x) = x + \alpha \bmod 1$. Fix an interval $I \subset [0, 1)$. Prove that

$$\limsup_{n\to\infty} \nu(I \cap (T^n)^{-1}I \cap \cdots \cap (T^{dn})^{-1}I) = \nu(I)$$

for all $d \in \mathbb{N}$.

2.7 Fix $\alpha \in \mathbb{R}$ irrational and an interval $A \subset [0,1)$. Let $T : [0,1) \to [0,1)$ be the map $T(x) = x + \alpha \bmod 1$.

(a) Verify that the set
$$R = \{n \in \mathbb{N} : T^n(0) \in A\}$$
has positive density.

(b) Prove that R contains arithmetic progressions of all lengths.

2.8 Verify that the set of primes does not have positive density by proving the following upper bound of Chebyshev. Define ϑ by
$$\vartheta(x) = \sum_{p \le x} \log p$$
where the sum is over all primes not larger than $x \in \mathbb{N}$.

(a) Prove for all $m \in \mathbb{N}$ that
$$\prod_{m \le p \le 2m} p \le \binom{2m}{m} \le 2^{2m}$$
where the product is restricted to p prime.

(b) Every $x \in \mathbb{N}$ lies in some interval $(2^i, 2^{i+1}]$. Combine with an estimate for $\vartheta(2m) - \vartheta(m)$ using the previous step to show that $\vartheta(x) \le 2x$ for all $x \in \mathbb{N}$.

(c) Write $\pi(x)$ for the number of primes less than or equal to $x \in \mathbb{R}$. Prove that
$$\pi(x) \le \pi(\sqrt{x}) + \frac{1}{\log \sqrt{x}} \sum_{\sqrt{x} < p \le x} \log p$$
for all x.

(d) Trivially estimate $\pi(\sqrt{x})$ and combine with (b) to deduce that $\pi(x) \log x \le 4x$.

(e) Use this to prove the primes do not have positive density.

2.4 The Oppenheim conjecture

2.9 A real number a is **badly approximable** if there is a constant $c > 0$ such that
$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}$$
for all $\frac{p}{q} \in \mathbb{Q}$.

(a) Prove (or recall from Lesson 8) that the golden ration $\phi = (1 + \sqrt{5})/2$ is badly approximable.

(b) Prove for the quadratic form $Q(x, y) = x^2 - \phi^2 y^2$ that $Q(\mathbb{Z}^2)$ is not dense in $\mathbb{R}$.

2.10 Fix the quadratic form $P(x, y, z) = x^2 + y^2 - z^2$.

(a) Describe the set $\mathsf{SO}(P)$ of matrices $A \in \mathsf{SL}(3, \mathbb{R})$ with the property that $P(Av) = P(v)$ for all $v \in \mathbb{R}^3$.

Fix a quadratic form Q in three variables that satisfies the hypothesis of the Oppenheim conjecture.

(b) Prove that there is $\lambda \in \mathbb{R}$ and $g \in \mathsf{SL}(3, \mathbb{R})$ such that $Q = \lambda(P \circ g)$.

(c) Prove that $\mathsf{SO}(Q) = g\mathsf{SO}(P)g^{-1}$.

(d) Prove that if $\mathsf{SO}(Q)\mathbb{Z}^3$ is dense in $\mathbb{R}^3$ then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

(e) Describe how $\mathsf{SL}(3, \mathbb{R})/\mathsf{SL}(3, \mathbb{Z})$ is the space of **unimodular lattices** i.e. lattices in $\mathbb{R}^3$ of the form $g\mathbb{Z}^3$ for some $g \in \mathsf{SL}(3, \mathbb{R})$.

(f) Prove that if the $\mathsf{SO}(P)$ orbit of the coset $g\mathsf{SL}(3, \mathbb{Z})$ is dense in $\mathsf{SL}(3, \mathbb{R})/\mathsf{SL}(3, \mathbb{Z})$ then $P(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.

Lesson 10: Continued fractions and Diophantine approximation

1 Material

Diophantine Approximation is a branch of number theory concerned with the approximation of real numbers with rational numbers. In Lesson 8, we discussed the existence of rational solutions p/q to the inequality

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}, \tag{1}$$

which measures how well p/q approximates x as a function of the height q of the rational number. Dirichlet's theorem told us that for all irrational numbers x, there are infinitely many rational numbers p/q satisfying (1), but the theorem didn't characterize such rational numbers or give us an efficient way to find them.

Thus, we were left with the questions:

(I) Which rational numbers satisfy inequalities such as the one in (1)?

(II) How do we find such rational approximations?

The theory of continued fractions provides a remarkably satisfactory answer to both of these questions, as we will see in this lesson.

From now on, whenever we write p/q, it is tacitly assumed that $q > 0$ and p and q are coprime so that p/q is in lowest terms. The number $x \in \mathbb{R}$ will always be irrational and will have continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$. The n^{th} convergent of the continued fraction $[a_0; a_1, a_2, \ldots]$ will always be denoted by p_n/q_n.

1.1 Continued fraction convergents as good rational approximations

The first theorem tells us how well continued fraction convergents approximate real numbers.

Theorem 25. *For all $n \geq 1$,*

$$\frac{1}{(a_{n+1}+2)q_n^2} < \frac{1}{q_n(q_n+q_{n+1})} < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1}q_n^2} \tag{2}$$

Proof. The first and fourth inequalities follow easily from the fact that $q_{n+1} = a_{n+1}q_n + q_{n-1}$ and the fact that the sequence $n \mapsto q_n$ is positive and increasing.

For the second and third inequalities, we will use the fact that $|p_{n+1}q_n - p_nq_{n+1}| = 1$ and the fact that the sequence

$$\frac{p_n}{q_n}, \quad \frac{p_{n+2}}{q_{n+2}} = \frac{p_n + a_{n+2}p_{n+1}}{q_n + a_{n+2}q_{n+1}}, \quad x, \quad \frac{p_{n+1}}{q_{n+1}}$$

is either increasing or decreasing.

The third inequality follows immediately on noting that

$$\left|x - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_nq_{n+1}}.$$

To see the second inequality, we will show first that $(p_n + p_{n+1})/(q_n + q_{n+1})$ falls between p_n/q_n and x. The convergent p_{n+2}/q_{n+2} is gotten by taking the mediant[5] of p_n/q_n with p_{n+1}/q_{n+1} exactly a_{n+2} many times, each successive mediant lying between the previous one and p_{n+1}/q_{n+1}. Since $(p_n+p_{n+1})/(q_n+q_{n+1})$ is the first of these mediants, it must fall between p_n/q_n and x. The second inequality now follows easily by noting that

$$\left|x - \frac{p_n}{q_n}\right| > \left|\frac{p_n + p_{n+1}}{q_n + q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n(q_n+q_{n+1})}.$$

This concludes the proof of the theorem. $\square$

[5]The **mediant** of a/b and c/d, where $b, d > 0$, is $(a+c)/(b+d)$; it is an exercise to show that it lies between a/b and c/d.

A basic conclusion from Theorem 25 is that *continued fraction convergents of any real number x provide good rational approximations to x.* Indeed, since $a_{n+1} \geq 1$, we see from (2) that every convergent satisfies the inequality from Dirichlet's theorem in (1). Thus, finding good rational approximations to an irrational number can be done using the Euclidean Algorithm via continued fractions!

Another conclusion from Theorem 25 is that *the partial quotients $a_1, a_2, \ldots,$ are intimately related to the degree to which the n^{th} convergent approximates x.* Indeed, it follows from (2) that a_{n+1} is determined to within one integer by the quantity $1/(q_n\|q_n x\|)$. Thus, the partial quotient a_{n+1} is very large if and only if the difference $|x - p_n/q_n|$ is very small. This leads to a very natural description of the set of badly approximable numbers that is explored in the exercises.

Finally, the following theorem is a strengthening of Dirichlet's theorem that follows quickly as a corollary of Theorem 25.

Theorem 26. *For all irrational numbers x and all $n \in \mathbb{N}$, at least one of p_n/q_n or p_{n+1}/q_{n+1} satisfies the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

In particular, there are infinitely many rational solutions to this inequality.

Proof. The real number x falls between p_n/q_n and p_{n+1}/q_{n+1}. If the conclusion of the theorem did not hold, then

$$\frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2} \leq \left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}}.$$

This simplifes to the inequality $(q_n - q_{n+1})^2 \leq 0$, which fails since $q_n < q_{n+1}$. This contradiction means in fact the conclusion must hold. $\square$

Going further, it is a famous theorem of Borel that at least one of every three consecutive continued fraction convergents satisfies

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

Thus, continued fraction convergents also give us an efficient way to find solutions to Hurwitz' strengthening of Dirichlet's approximation theorem that we saw in Lesson 8.

1.2 Continued fraction convergents as the best rational approximations

Now that we have seen that continued fraction convergents provide *good* rational approximates, we take up the matter of whether they are the *best* rational approximates.

Call p/q a **best approximate to** x if it minimizes the distance to x amongst all rational numbers with height at most q. We will see in this section that

(I) all continued fraction convergents of x are best approximates to x;

(II) not all best approximates to x are continued fraction convergents of x;[6]

(III) if p/q is a very good approximate to x in the sense that $|x - p/q| < 1/(2q^2)$, then p/q is a continued fraction convergent of x.

The statement in (II) indicates some subtlety in the matter, and it is this subtlety that causes the statement of the next theorem to look, at first sight, unnatural.

Theorem 27 (Law of Best Approximates)**.** *Of all quantities $|qx - p|$ with $1 \leq q < q_{n+1}$ and $p \in \mathbb{Z}$, the quantity $|q_n x - p_n|$ is nearest to zero.*

You will prove Theorem 27 in the exercises. Before we derive the promised corollaries from it, let us see how it can be used to characterize the continued fraction convergents of an irrational number x. Call an increasing sequence of positive integers $(n_i)_{i\in\mathbb{N}}$ a **sequence of records for** x if

[6] It is true, however, that every best approximate to x is derived from a convergent in some sense; see [Khi2] for more details.

- $n_1 = 1$;
- $\|n_1 x\| > \|n_2 x\| > \|n_3 x\| > \cdots$; and
- for all $n_i < n < n_{i+1}$, $\|nx\| > \|n_i x\|$.

The Law of Best Approximates gives us the following remarkable fact: *the sequence of denominators $(q_n)_{n\in\mathbb{N}}$ of convergents of x is a sequence of records for x.* The terminology comes from keeping track of the first n's that break the previous record by being closest to zero on the circle modulo 1. This characterizes continued fraction denominators, and hence numerators and partial quotients, as "record setting times" of the orbit of zero under a circle rotation.

Theorem 28. *Of all fractions p/q with $q \le q_n$, the convergent p_n/q_n is nearest to x.*

This theorem follows quickly from Theorem 27, as you will show in the exercises. It says that every continued fraction convergent of x is a best approximate to x. But it is instructive to see what it does not say: it does not say that every best approximate to x is a continued fraction convergent. You will find examples in the exercises.

Theorem 27 also allows us to show that the only very good rational approximations to x are its continued fraction convergents. The following theorem makes this precise.

Theorem 29 (Legendre's theorem)**.** *If*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}, \tag{3}$$

then p/q is a continued fraction convergent of x.

Again, you will work through a proof in the exercises. And again, it is instructive to understand what Legendre's theorem does not say: it does not say that all continued fraction convergents satisfy (3). In fact, as an exercise, you are tasked with finding some that do not.

1.3 The typical rate of approximation

The inequalities in Theorem 25 combine with the Ergodic Theorem to produce a marvelous result concerning the rate of convergence of continued fraction convergents to the typical real number.

Theorem 30 (Lévy-Khinchin theorem)**.** *For Lebesgue almost every $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log\left|x - \frac{p_n}{q_n}\right| = -\frac{\pi^2}{6\log 2}$$

Proof. We have from Theorem 25 that

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}. \tag{4}$$

Thus, we can estimate the distance between x and the n^{th} continued fraction convergent p_n/q_n as a function of the growth rate of the convergents' denominators.

It is a fact provable using the Ergodic Theorem [EW, Corollary 3.8] that for Lebesgue almost every $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12\log 2}.$$

Thus, for a typical real number, the sequence of continued fraction convergent denominators grows exponentially with base $e^{\pi^2/12\log 2} \approx 3.276$. The conclusion of the Lévy-Khinchin theorem follows by a short calculation from combining the inequalities in (4) with this fact. $\square$

The Lévy-Khinchin theorem says that for a typical real number x, there exists a sequence $(\epsilon_n)_{n=1}^{\infty}$ tending to zero such that for all $n \in \mathbb{N}$,

$$\left|x - \frac{p_n}{q_n}\right| = e^{(-\pi^2/6\log 2 + \epsilon_n)n}.$$

Thus, typically, for large n, the n^{th} partial quotient of x is at a distance of about $e^{-2.373n}$ from x. The number $e^{-\pi^2/6\log 2}$ is just less than $1/10$, so typically, for large n, the n^{th} partial quotient approximates x to within at least n many decimal places.

2 Problems

2.1 Best approximations to π

***2.1** Compute the first few continued fraction convergents to π. How do these compare to the infamous $22/7$?

2.2 Find a few best approximations to π that are not a continued fraction convergents of π.

2.3 Find some continued fraction convergents to π that do not satisfy the inequality in (3).

2.4 ([Loy]) Let $\varphi : \mathbb{N} \to (0,1)$ be a function. Show that there exists an irrational number ξ and infinitely many rationals p/q satisfying

$$\left|\xi - \frac{p}{q}\right| < \varphi(q).$$

Combine this theorem with Liouville's theorem to easily produce transcendental numbers.

2.2 Proof of Theorem 27, the Law of Best Approximates

This sequence of exercises comprises a complete proof of Theorem 27.

2.5 Argue that the following two statements suffice to prove the theorem.

(I) The sequence $n \mapsto |q_n x - p_n|$ is decreasing.

(II) If $p \in \mathbb{Z}$ and $q \geq 1$ minimize the expression $|q'x - p'|$ amongst all $p' \in \mathbb{Z}$ and $1 \leq q' \leq q$, then p/q is a continued fraction convergent of x.

2.6 Use the inequalities in Theorem 25 to show that the sequence $n \mapsto |q_n x - p_n|$ is decreasing. This establishes (I).

Establishing (II) is a bit more challenging. Suppose that $p \in \mathbb{Z}$ and $q \geq 1$ minimize the expression $|q'x - p'|$ amongst all $p' \in \mathbb{Z}$ and $1 \leq q' \leq q$. Suppose also that p/q is not a convergent of x. We aim to reach a contradiction.

2.7 Show that if $p/q < a_0$, then $|x - a_0| < |qx - p|$, contradicting the assumption on p and q.

2.8 Show that if $p/q > p_1/q_1$, then $|x - a_0| < |qx - p|$, contradicting the assumption on p and q. (Hint: Show that $|x - p/q| > |p_1/q_1 - p/q| \geq 1/(qq_1)$ and that $|x - a_0| \leq 1/a_1$, then combine these facts.)

2.9 Conclude from the previous two exercises and the alternating property of the convergents that p/q must lie between two convergents p_{m-1}/q_{m-1} and p_{m+1}/q_{m+1} for some $m \geq 0$.

2.10 Show that

$$\frac{1}{qq_{m-1}} \leq \left|\frac{p}{q} - \frac{p_{m-1}}{q_{m-1}}\right| \leq \left|\frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}}\right| = \frac{1}{q_{m-1}q_m},$$

and conclude that $q > q_m$.

2.11 Show that

$$\frac{1}{qq_{m+1}} \leq \left|\frac{p_{m+1}}{q_{m+1}} - \frac{p}{q}\right| \leq \left|x - \frac{p}{q}\right|,$$

and combine this with Theorem 25 to conclude that

$$|q_m x - p_m| \leq \frac{1}{q_{m+1}} \leq |qx - p|.$$

2.12 Combine **2.10** and **2.11** to reach a contradiction with the assumptions on p and q and conclude the proof of Theorem 27.

2.3 Proof of Theorem 28

2.13 Suppose that p/q is the rational number with $q \leq q_n$ that is nearest to x. Show that if $q = q_n$, then $p/q = p_n/q_n$.

2.14 Show that if $q < q_n$, then $|x - p/q| < |x - p_n/q_n|$. Conclude that $|qx - p| < |q_n x - p_n|$.

2.15 Combine the previous exercise with Theorem 27 to reach a contradiction and conclude the proof of Theorem 28.

2.4 Proof of Theorem 29, Legendre's theorem

2.16 Argue using Theorem 27 that Legendre's theorem follows from the statement: if $|x - p/q| < 1/(2q^2)$ and the rational r/s is such that $|sx - r| < |qx - p|$, then $s > q$.

2.17 Suppose that $|x - p/q| < 1/(2q^2)$ and that the rational r/s is such that $|sx - r| < |qx - p|$. Show that $|x - r/s| < 1/(2qs)$.

2.18 Use the triangle inequality and the fact that $r/s \neq p/q$ to show that

$$\frac{1}{qs} < \left|\frac{r}{s} - \frac{p}{q}\right| < \frac{1}{2q^2} + \frac{1}{2qs}.$$

2.19 Conclude from the previous problem that $s > q$, concluding the proof of Legendre's theorem.

2.5 Badly approximable numbers

Recall from Lesson 8 that an irrational number x is **badly approximable** if there exists $c > 0$ such that for all p/q,

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

2.20 Use Theorem 25 to show that x is badly approximable if and only if it has "bounded partial quotients," that is, the sequence of its partial quotients is bounded.

2.21 It is a fact that a real number x is a quadratic irrational if and only if the sequence of its partial quotients is eventually periodic. Observe that it follows immediately from this fact that all quadratic irrationals are badly approximable.

2.22 Show that the set of badly approximable numbers is uncountable.

2.23 Use the Ergodic Theorem to show that the set of badly approximable numbers in $[0, 1]$ has zero measure with respect to the Gauss measure. Conclude that the set of badly approximable numbers has zero Lebesgue measure.

2.24 Show that the set of pairs (x, y) that do not satisfy the Littlewood conjecture (from Lesson 8) has zero Lebesgue measure in the plane.

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