ADDITIONAL EXERCISES: TUESDAY

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(1) Show $|t\zeta(1+it)| \geq 1$ for all $t \in \mathbb{R}$.
(2) Is $\mathrm{Re}(\zeta(1+it)) > 0$ for $t \neq 0$?
(3) Explain the "paradox": Define

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \qquad \text{for } \sigma > 1,$$

$$F(s) = \prod_p \frac{1}{1+p^{-s}} \qquad \text{for } \sigma > 1.$$

Since these are absolutely convergent, we can multiply together to show

$$\zeta(s)F(s) = \prod_p \frac{1}{1-p^{-2s}} = \zeta(2s).$$

We have two meromorphic function that agree for $\sigma > 1$, so their analytic continuations agree

$$\zeta(s)F(s) = \zeta(2s).$$

What happens on the critical line?

$$\zeta(\frac{1}{2}+it)F(\frac{1}{2}+it) = \zeta(1+2it) \neq 0.$$

Thus, there are no zeros on the critical line. Explain.