Approximation of Riemann'z Zeta Function by Finite Dirichlet Series. I

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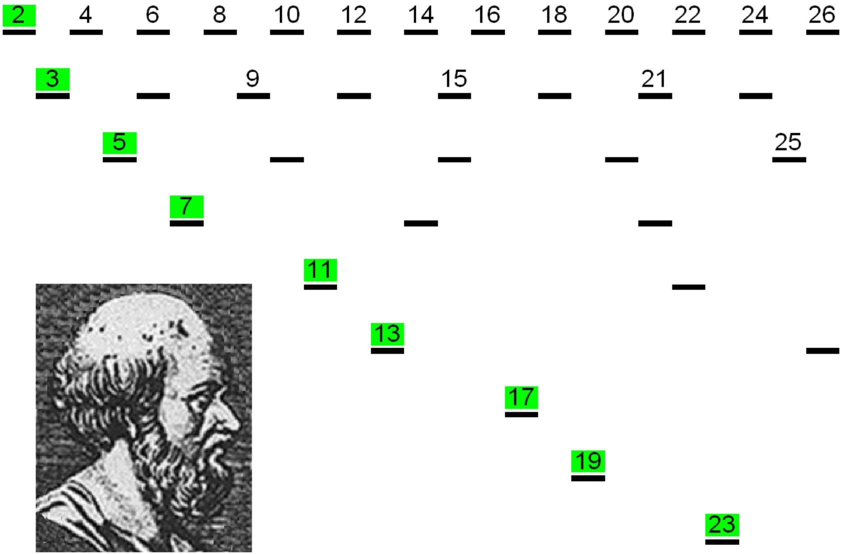
http://logic.pdmi.ras.ru/~yumat/personaljournal/finitedirichlet

Sieve of Eratosthenes (276–194 B. C.)



Leonhard Euler (1707–1783)

Euler identity:

$$1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots = \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}$$

The infinitude of prime numbers

Theorem (Euclid). *There are infinitely many prime numbers.*

New proof (Euler). If the number of primes would be finite, then the (divergent) harmonic series would have finite value:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots=\prod_{p \text{ is prime}}\frac{1}{1-\frac{1}{p}}$$

Basel Problem (Pietro Mengoli, 1644)

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = ?$$

Leonhard Euler:

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \zeta(2) &= 1.64493406684822644\ldots \\ \frac{\pi^2}{6} &= 1.64493406684822644\ldots \end{aligned}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

In 1735 Euler gave his first “proof” of this equality

Another value of $\zeta(s)$ given by EULER

$$\zeta(0) = 1^0 + 2^0 + 3^0 + \cdots = 1 + 1 + 1 + \ldots = -\frac{1}{2}$$

$$\begin{aligned}
\eta(s) &= (1 - 2 \cdot 2^{-s})\zeta(s) \\
&= (1 - 2 \cdot 2^{-s})(1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \ldots) \\
&= 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + \ldots \\
&\quad - 2 \cdot 2^{-s} \qquad - 2 \cdot 4^{-s} - \ldots \\
&= 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \ldots
\end{aligned}$$

The alternating Dirichlet series

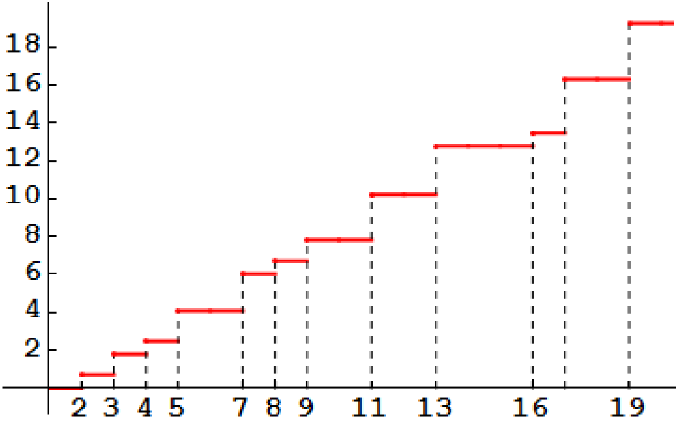
$$1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \ldots$$

converges for $s > 0$.

$$1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}$$

Chebyshev's function $\psi(x)$

$$\psi(x) \quad = \sum_{\substack{q \le x \\ q \text{ is a power} \\ \text{of a prime } p}} \ln(p) = \ln(\mathrm{LCM}(1, 2, ..., \lfloor x \rfloor))$$



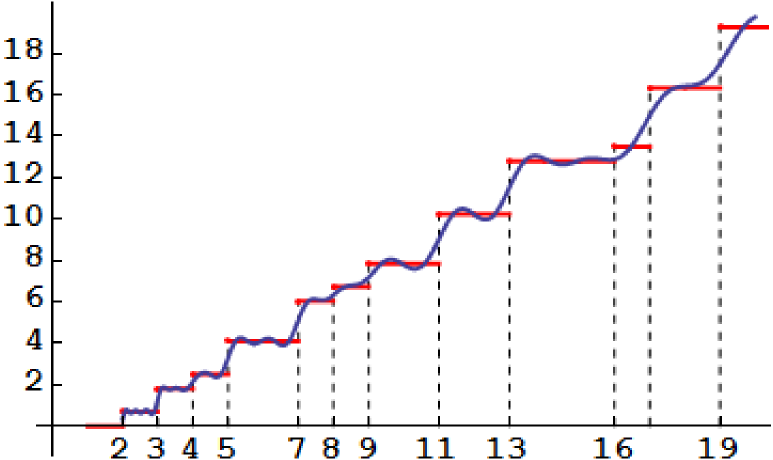
Theorem of von Mangoldt

Теорема (Hans Carl Fridrich von Mangoldt [1895]).

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \ln(2\pi)$$

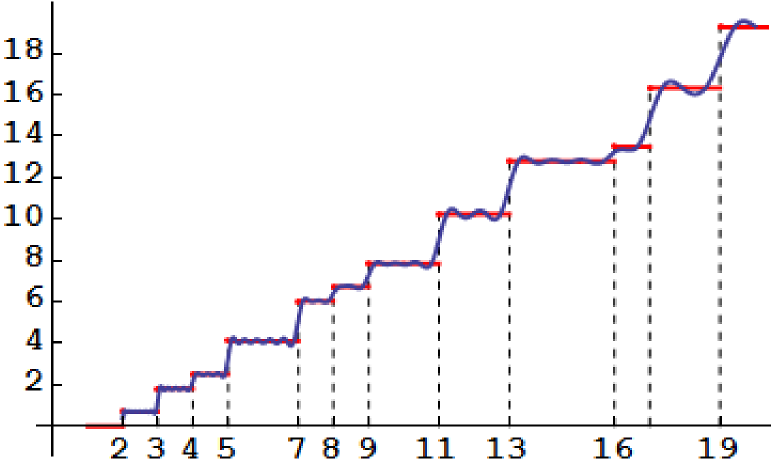
Theorem of von Mangoldt

$$\psi(x) \qquad x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<50}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



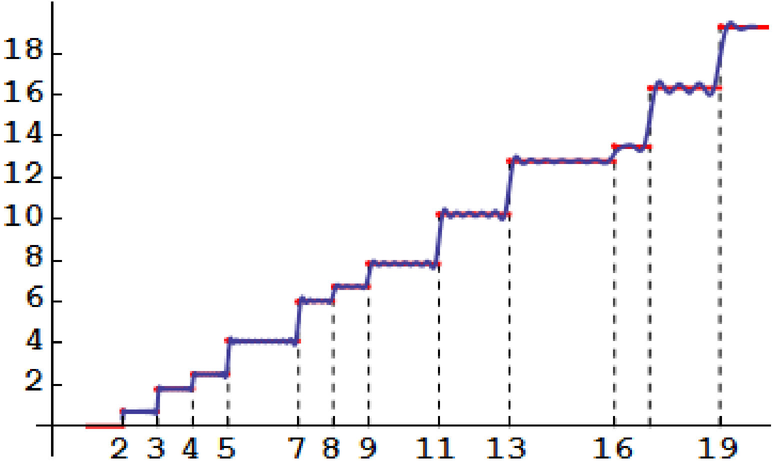
Theorem of von Mangoldt

$$\psi(x) \qquad x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<100}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



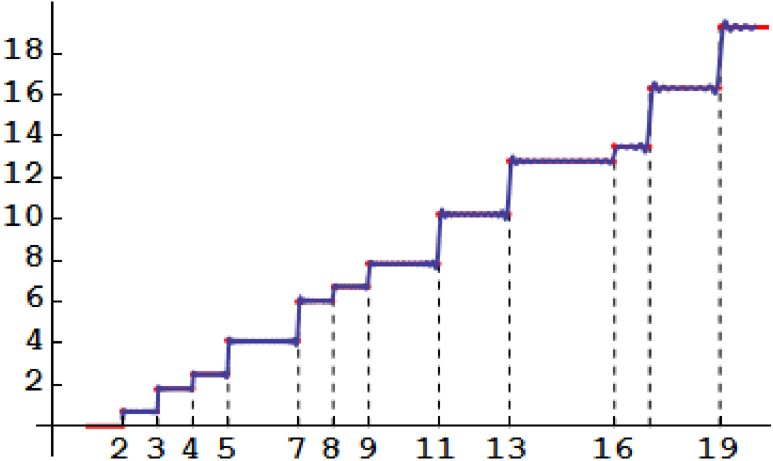
Theorem of von Mangoldt

$$\psi(x) \qquad x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<200}} \frac{x^{\rho}}{\rho} - \ln(2\pi)$$



Theorem of von Mangoldt

$$\psi(x) \qquad x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<400}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



Approximations by Dirichlet series

$$\zeta(s) = 1^{-s} + 2^{-s} + \cdots + n^{-s} \ldots$$

$$1^{-s} + 2^{-s} + \cdots + N^{-s}$$

$$\Delta_N(s) = \delta_{N,1} \qquad 1^{-s} + \delta_{N,2} 2^{-s} + \cdots + \delta_{N,N} N^{-s} \qquad \delta_{N,1} = 1$$

$$\cdots = \zeta(\overline{\rho_3}) = \zeta(\overline{\rho_2}) = \zeta(\overline{\rho_1}) = 0 = \zeta(\rho_1) = \zeta(\rho_2) = \zeta(\rho_3) = \ldots$$

$$\rho_n = \frac{1}{2} + \mathrm{i}\gamma_n \qquad 0 < \gamma_1 < \gamma_2 < \gamma_3 \ldots$$

$$N = 2M + 1$$

$$\Delta_N(\overline{\rho_M}) = \cdots = \Delta_N(\overline{\rho_1}) = 0 = \Delta_N(\rho_1) = \cdots = \Delta(\rho_M)$$

Trivial case of polynomials

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad a_0 \neq 0$$

$$P(x_1) = \cdots = P(x_n) = 0 \qquad i \neq j \Rightarrow x_i \neq x_j$$

$$P(z) = a_0 \left(1 - \frac{z}{x_1}\right) \ldots \left(1 - \frac{z}{x_n}\right)$$

Case of Taylor series of an entire function

$$A(z) = a_0 + a_1 z + \dots + a_n z^n + \dots \qquad a_0 \neq 0$$

$$0 = A(x_1) = \dots = A(x_n) = \dots \qquad i \neq j \Rightarrow x_i \neq x_j$$

$$A(z) = a_0 \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right)$$

$$\begin{aligned} A_N(z) &= a_0 \prod_{k=1}^{N} \left(1 - \frac{z}{x_k}\right) \\ &= a_{N,0} + a_{N,1} z + \dots + a_{N,N} z^N \end{aligned}$$

$$a_{N,n} \xrightarrow[N \to \infty]{} a_n$$

Approximations by Dirichlet series (cont.)

$$\Delta_N(s) = 1^{-s} + \delta_{N,2} 2^{-s} + \cdots + \delta_{N,N} N^{-s}$$

$$N = 2M + 1$$

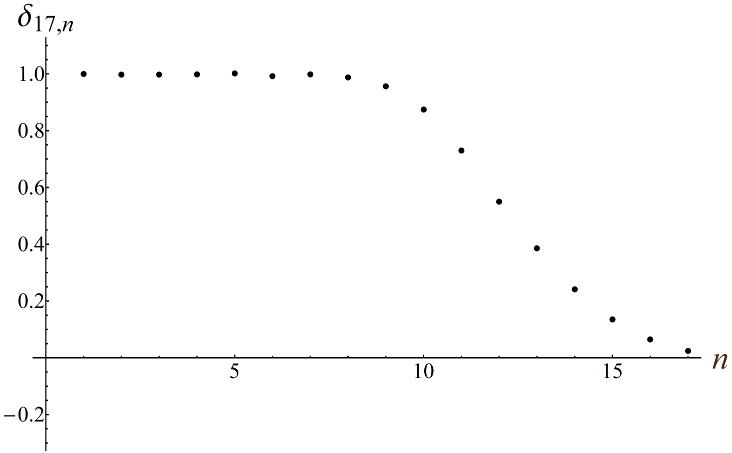
$$\Delta_N(\overline{\rho_M}) = \cdots = \Delta_N(\overline{\rho_1}) = 0 = \Delta_N(\rho_1) = \cdots = \Delta(\rho_M)$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\overline{\rho_1}} & n^{-\rho_1} & \dots & n^{-\overline{\rho_M}} & n^{-\rho_M} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} & N^{-s} \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} n^{-s}$$

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}}$$

Question. $\delta_{N,n} \xrightarrow[N\to\infty]{?} 1$

Coefficients $\delta_{17,n}$

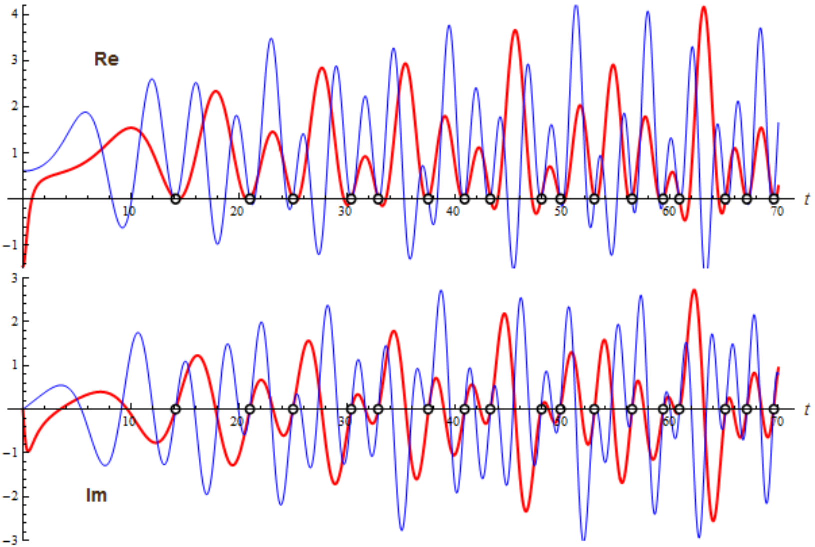


Question: $\delta_{N,n} \underset{N\to\infty}{\longrightarrow} 1$? Looks plausible...

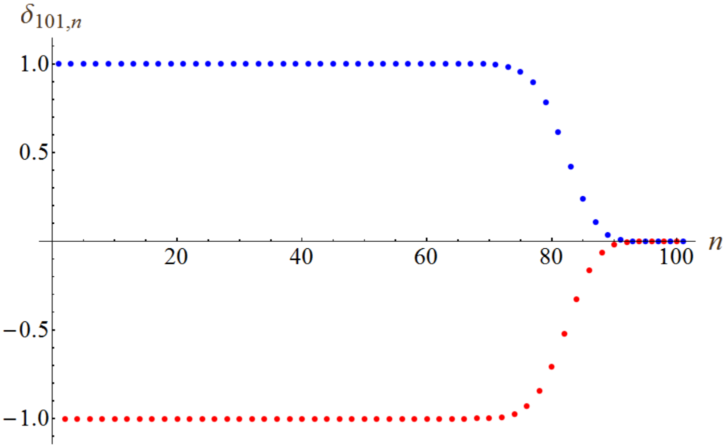
$$\Delta_{17}(s) = \sum_{n=1}^{17} \delta_{17,n} n^{-s} \leftrightharpoons \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$$

$\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Delta_{17}(\frac{1}{2} + \mathrm{i}t) = \sum_{n=1}^{17} \delta_{17,n} n^{-\frac{1}{2} - \mathrm{i}t}$

$\zeta(\frac{1}{2}+\mathrm{i}t)$ and $\Delta_{101}(\frac{1}{2}+\mathrm{i}t)=\sum_{n=1}^{101}\delta_{101,n}n^{-\frac{1}{2}-\mathrm{i}t}$



Coefficients $\delta_{101,n}$, red for even n, blue for odd n



Case of Taylor series of a meromorphic function

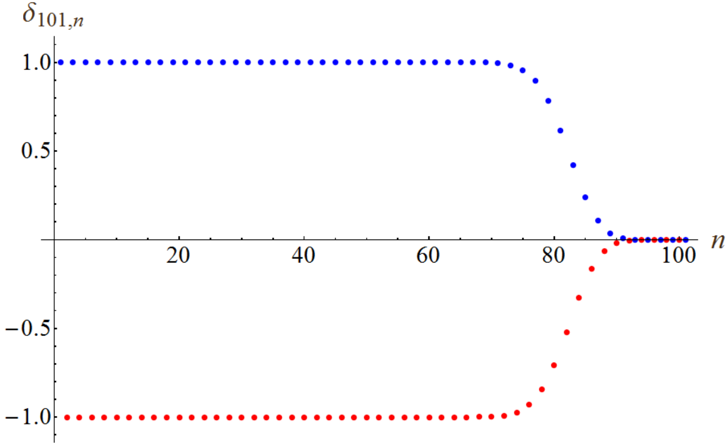
$$A(z) = a_0 + a_1 z + \cdots + a_n z^n + \ldots = \frac{a_0 \prod_{k=1}^{\infty}\left(1 - \frac{z}{x_k}\right)}{\prod_{k=1}^{\infty}\left(1 - \frac{z}{y_k}\right)}$$

$$A_N(z) \;\; = \;\; a_0 \prod_{k=1}^{N}\left(1 - \frac{z}{x_k}\right) = a_{N,0} + a_{N,1} z + \cdots + a_{N,N} z^N$$

$$a_{N,n} \xrightarrow[N\to\infty]{?} a_n \qquad\qquad a_{N,n} \xrightarrow[N\to\infty]{} b_n$$

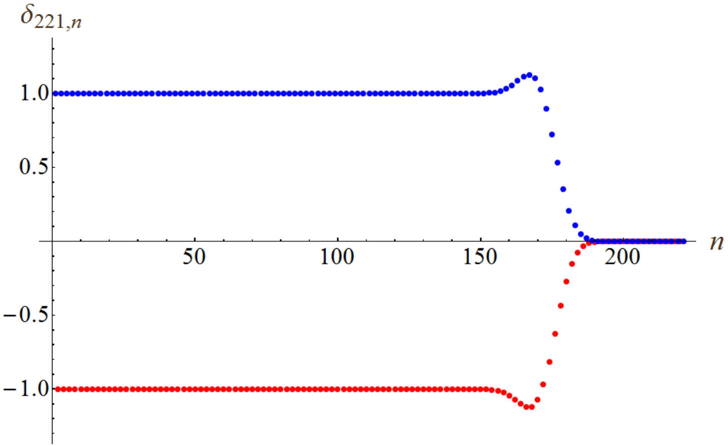
$$b_0 + b_1 z + \cdots + b_n z^n + \ldots = a_0 \prod_{k=1}^{\infty}\left(1 - \frac{z}{x_k}\right) = A(z) \prod_{k=1}^{\infty}\left(1 - \frac{z}{y_k}\right)$$

New conjecture

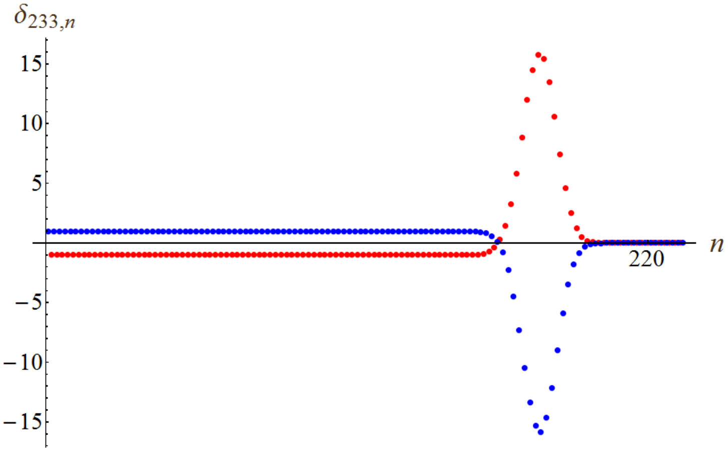


Conjecture: $\delta_{N,n} \xrightarrow[N\to\infty]{} (-1)^{n+1}$

Coefficients $\delta_{221,n}$, red for even n, blue for odd n



Coefficients $\delta_{233,n}$, red for even n, blue for odd n

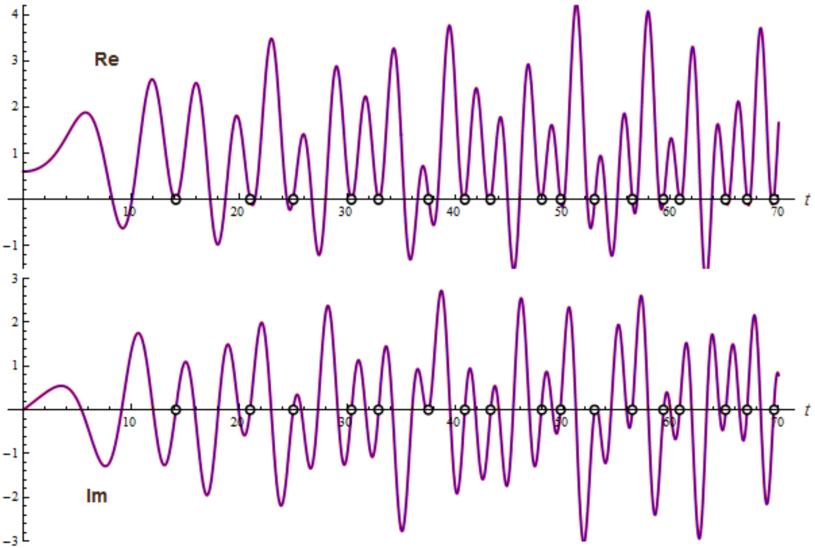


Implication of the conjecture

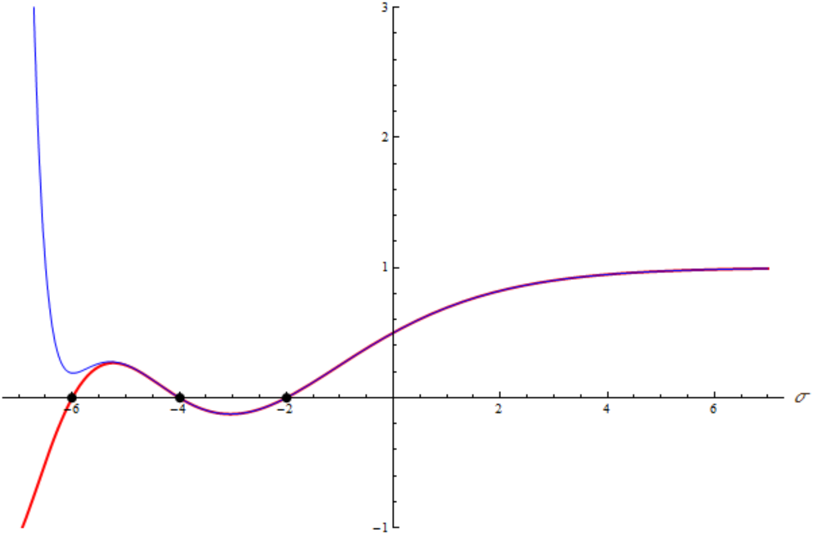
Conjecture: $\delta_{N,n} \underset{N\to\infty}{\longrightarrow} (-1)^{n+1}$

$$\Delta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} \leftrightharpoons \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = \eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s)$$

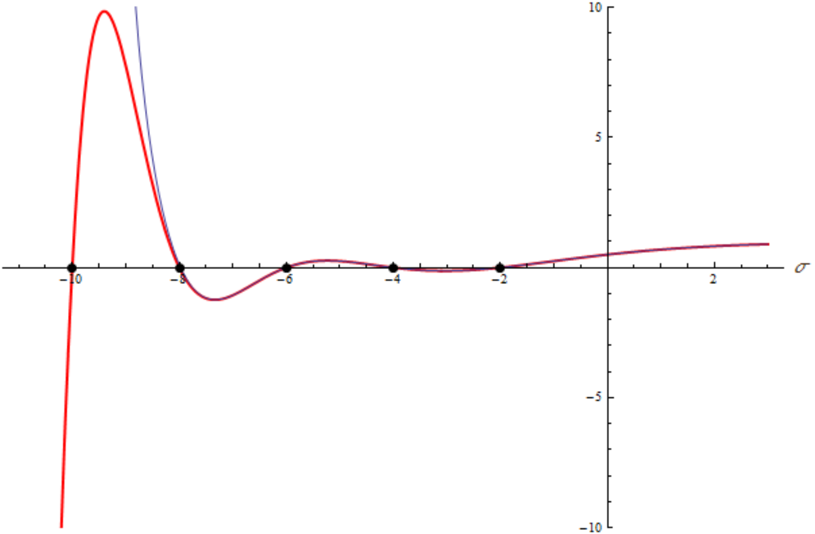
$\eta(\frac{1}{2} + \mathrm{i}t) = (1 - 2 \cdot 2^{-\frac{1}{2} - \mathrm{i}t})\zeta(\frac{1}{2} + \mathrm{i}t)$ and $\Delta_{101}(\frac{1}{2} + \mathrm{i}t)$



$\eta(\sigma) = (1 - 2 \cdot 2^{-\sigma})\zeta(\sigma)$ and $\Delta_{101}(\sigma)$



$\eta(\sigma) = (1 - 2 \cdot 2^{-\sigma})\zeta(\sigma)$ and $\Delta_{121}(\sigma)$



$\eta(\frac{1}{2}+\mathrm{i}t)=\zeta(\frac{1}{2}+\mathrm{i}t)$ and $\Delta_{17}(\frac{1}{2}+\mathrm{i}t)=\sum_{n=1}^{17}\delta_{17,n}n^{-\frac{1}{2}-\mathrm{i}t}$

Non-trivial zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$\begin{aligned}
0 &= \Delta_N(\rho_{M+1} - 5.154\ldots\cdot 10^{-1157} + 1.120\ldots\cdot 10^{-1156}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+201} - 4.922\ldots\cdot 10^{-890} - 9.995\ldots\cdot 10^{-891}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+401} - 3.159\ldots\cdot 10^{-735} - 2.750\ldots\cdot 10^{-735}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+601} + 8.765\ldots\cdot 10^{-619} + 4.575\ldots\cdot 10^{-618}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+801} + 2.075\ldots\cdot 10^{-524} + 1.197\ldots\cdot 10^{-524}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1001} + 1.980\ldots\cdot 10^{-447} - 3.397\ldots\cdot 10^{-448}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1201} - 1.034\ldots\cdot 10^{-381} - 1.354\ldots\cdot 10^{-382}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1401} + 1.466\ldots\cdot 10^{-326} - 1.835\ldots\cdot 10^{-326}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1601} + 2.281\ldots\cdot 10^{-278} - 3.603\ldots\cdot 10^{-278}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+1801} - 7.799\ldots\cdot 10^{-237} - 3.726\ldots\cdot 10^{-237}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2001} + 5.921\ldots\cdot 10^{-201} - 6.855\ldots\cdot 10^{-201}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2201} + 8.049\ldots\cdot 10^{-170} + 1.359\ldots\cdot 10^{-169}\mathrm{i}) \\
0 &= \Delta_N(\rho_{M+2401} - 2.001\ldots\cdot 10^{-142} - 7.023\ldots\cdot 10^{-142}\mathrm{i})
\end{aligned}$$

Trivial zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$
\begin{aligned}
0 &= \Delta_N(-2 - 1.884\ldots \cdot 10^{-1510}) \\
0 &= \Delta_N(-4 + 2.013\ldots \cdot 10^{-1504}) \\
0 &= \Delta_N(-6 - 1.158\ldots \cdot 10^{-1498}) \\
0 &= \Delta_N(-8 + 4.508\ldots \cdot 10^{-1493}) \\
0 &= \Delta_N(-10 - 1.316\ldots \cdot 10^{-1487}) \\
0 &= \Delta_N(-12 + 3.066\ldots \cdot 10^{-1482}) \\
0 &= \Delta_N(-14 - 5.931\ldots \cdot 10^{-1477}) \\
0 &= \Delta_N(-16 + 9.796\ldots \cdot 10^{-1472}) \\
0 &= \Delta_N(-18 - 1.410\ldots \cdot 10^{-1466}) \\
0 &= \Delta_N(-20 + 1.797\ldots \cdot 10^{-1461}) \\
0 &= \Delta_N(-22 - 2.054\ldots \cdot 10^{-1456}) \\
0 &= \Delta_N(-24 + 2.126\ldots \cdot 10^{-1451})
\end{aligned}
$$

Case of Taylor series of a meromorphic function (repeated)

$$A(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots = \frac{a_0 \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right)}{\prod_{k=1}^{\infty} \left(1 - \frac{z}{y_k}\right)}$$

$$A_N(z) \quad = \quad a_0 \prod_{k=1}^{N} \left(1 - \frac{z}{x_k}\right) = a_{N,0} + a_{N,1} z + \cdots + a_{N,N} z^N$$

$$a_{N,n} \xrightarrow[N\to\infty]{?} a_n \qquad\qquad a_{N,n} \xrightarrow[N\to\infty]{} b_n$$

$$b_0 + b_1 z + \cdots + b_n z^n + \cdots = a_0 \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) = A(z) \prod_{k=1}^{\infty} \left(1 - \frac{z}{y_k}\right)$$

Extra zeroes for $M = 1550$, $N = 2M + 1 = 3101$

$$\Delta_N(s) \leftrightharpoons \eta(s) \quad = \quad 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \cdots = (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$1 - 2 \cdot 2^{-s} = 0 \quad \Longleftrightarrow \quad s = s_k = 1 + \frac{2\pi k}{\ln(2)}\mathrm{i}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\begin{aligned}
0 &= \Delta_N(s_{50} - 5.481\ldots \cdot 10^{-133} - 5.546\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{100} - 1.109\ldots \cdot 10^{-132} - 1.306\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{150} - 5.743\ldots \cdot 10^{-133} + 5.543\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{200} - 6.157\ldots \cdot 10^{-136} + 2.613\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{250} - 5.220\ldots \cdot 10^{-133} - 5.537\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{300} - 1.108\ldots \cdot 10^{-132} - 3.917\ldots \cdot 10^{-134}\mathrm{i}) \\
0 &= \Delta_N(s_{350} - 6.004\ldots \cdot 10^{-133} + 5.528\ldots \cdot 10^{-133}\mathrm{i}) \\
0 &= \Delta_N(s_{400} - 2.461\ldots \cdot 10^{-135} + 5.220\ldots \cdot 10^{-134}\mathrm{i})
\end{aligned}$$

Zeta zeroes are very knowledgable

Zeta zeroes "know about"

- the initial trivial zeroes
- other non-trivial zeroes
- the pole of the zeta function via the zeroes of the factor $1 - 2 \cdot 2^{-s}$ cancelling the pole

Zeroes from Euler Product

$$\begin{aligned}\zeta(s) &= 1^{-s} + 2^{-s} + \cdots + k^{-s} + \ldots \\ &= \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}\end{aligned}$$

$$\begin{aligned}\zeta_2(s) &= \prod_{\substack{p \text{ is prime} \\ p \neq 2}} \frac{1}{1 - p^{-s}} \\ &= 1^{-s} + 3^{-s} + \cdots + (2k+1)^{-s} + \ldots \\ &= L(2, \chi_1, s) \\ &= (1 - 2^{-s})\zeta(s)\end{aligned}$$

An Example

$$\zeta_2(s) = (1 - 2^{-s})\zeta(s)$$

Let us take 100 (pairs of conjugate) zeros of the zeta function,

$$\frac{1}{2} \pm \mathrm{i}\gamma_k, \qquad k = 1, \dots, 100,$$

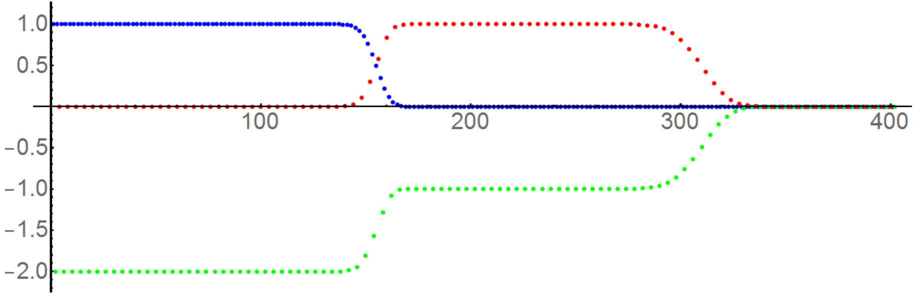
and 201 zeros of $1 - 2^{-s}$,

$$0, \pm\frac{2\pi\mathrm{i}}{\ln(2)}, \dots, \pm 100\frac{2\pi\mathrm{i}}{\ln(2)}$$

calculate corresponding 402 determinants of size 401 and normalize them getting numbers

$$\delta_{2,200,402,1}, \dots, \delta_{2,200,402,402}$$

$\delta_{2,200,402,n}$ for $\zeta_2(s) = (1 - 2^{-s})\zeta(s)$



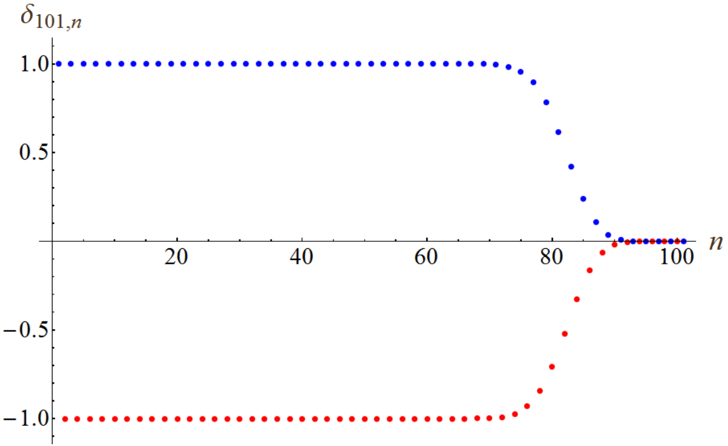
Blue for $n \equiv 1 \pmod 2$, Red for $n \equiv 0 \pmod 4$, Green for $n \equiv 2 \pmod 4$

$n < 120 \Rightarrow$ the δ's are close to the coefficients of $(1 - 2^{-s})(1 - 2 \cdot 2^{-s})\zeta(s)$

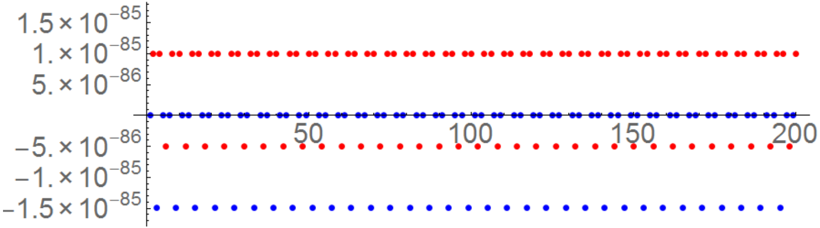
$190 < n < 290 \Rightarrow$ the δ's are close to the coefficients of $-2^{-s}(1 - 2 \cdot 2^{-s})\zeta(s)$

$330 < n \Rightarrow$ the δ's are very small

Coefficients $\delta_{101,n}$, red for even n, blue for odd n (repeated)



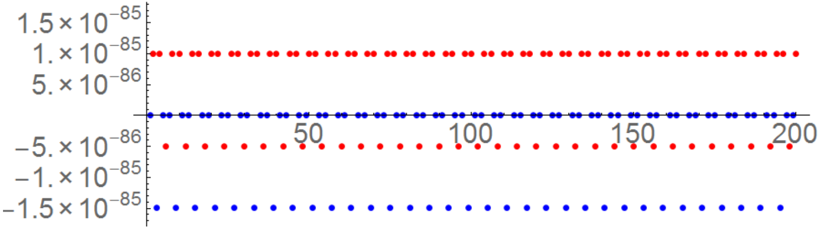
$\delta_{2001,n} - (-1)^{n+1}$ for odd n and for even n



$$\delta_{2001,2} + 1 = 9.93613.. \times 10^{-86} \qquad \delta_{2001,3} - 1 = -1.49042.. \times 10^{-85}$$

$$\frac{\delta_{2001,3} - 1}{\delta_{2001,2} + 1} = -\frac{3}{2} - 1.021969.. \times 10^{-114}$$

$\delta_{2001,n} - (-1)^{n+1}$ for odd n and for even n



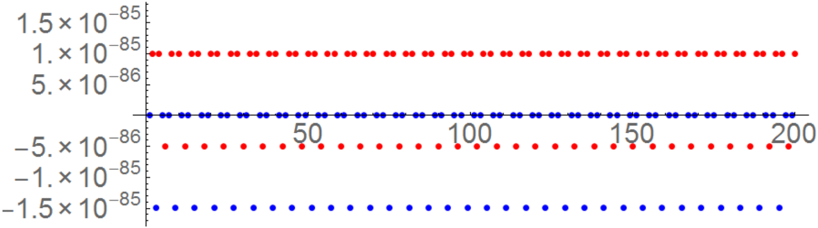
$$\delta_{2001,4} + 1 = 9.93613.. \times 10^{-86} \qquad \delta_{2001,2} + 1 = 9.93613.. \times 10^{-86}$$

$$\delta_{2001,4} - \delta_{2001,2} = -1.353923.. \times 10^{-199}$$

$$\frac{\delta_{2001,3} - 1}{\delta_{2001,2} + 1} = -\frac{3}{2} - 1.021969.. \times 10^{-114}$$

$$\frac{\delta_{2001,4} - \delta_{2001,2}}{3(\delta_{2001,2} + 1) + 2(\delta_{2001,3} - 1)} = -\frac{2}{3} - 1.688757.. \times 10^{-98}$$

$\delta_{2001,n} - (-1)^{n+1}$ for odd n and for even n



$$\delta_{2001,5} - 1 = 4.287089.. \times 10^{-297}$$

$$\frac{\delta_{2001,4} - \delta_{2001,2}}{3(\delta_{2001,2} + 1) + 2(\delta_{2001,3} - 1)} = -\frac{2}{3} - 1.688757.. \times 10^{-98}$$

$$\frac{\delta_{2001,5} - 1}{6(\delta_{2001,2} + 1) + 4(\delta_{2001,3} - 1) + 3(\delta_{2001,4} - \delta_{2001,2})} = -\frac{5}{12} - 1.1331.. \times 10^{-81}$$

Notation

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right)\delta_{N,m}$$

$$\mu(k) = \begin{cases} (-1)^m, & \text{if } k \text{ is the product of } m \text{ different primes} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mu_{N,1} &= 1 \\
\mu_{N,2} &= \delta_{N,2} - 1 \\
\mu_{N,3} &= \delta_{N,3} - 1 \\
\mu_{N,4} &= \delta_{N,4} - \delta_{N,2} \\
\mu_{N,5} &= \delta_{N,5} - 1 \\
\mu_{N,6} &= \delta_{N,6} - \delta_{N,3} - \delta_{N,2} + 1 \\
\mu_{N,7} &= \delta_{N,7} - 1 \\
\mu_{N,8} &= \delta_{N,8} - \delta_{N,4} \\
\mu_{N,9} &= \delta_{N,9} - \delta_{N,3}
\end{aligned}$$

Case $N = 2001$

$$
\begin{aligned}
\mu_{2001,2} &= -2 + 9.93613\ldots \cdot 10^{-86} \\
\mu_{2001,3} &= -1.49042\ldots \cdot 10^{-85} \\
\mu_{2001,4} &= -1.35392\ldots \cdot 10^{-199} \\
\mu_{2001,5} &= +4.28708\ldots \cdot 10^{-297} \\
\mu_{2001,6} &= -1.39904\ldots \cdot 10^{-377} \\
\mu_{2001,7} &= -8.46908\ldots \cdot 10^{-444} \\
\mu_{2001,8} &= -3.00897\ldots \cdot 10^{-499} \\
\mu_{2001,9} &= +2.56119\ldots \cdot 10^{-546} \\
\mu_{2001,10} &= +9.47153\ldots \cdot 10^{-587} \\
\mu_{2001,11} &= -2.22088\ldots \cdot 10^{-622} \\
\mu_{2001,12} &= +1.65346\ldots \cdot 10^{-653} \\
\mu_{2001,13} &= -1.33219\ldots \cdot 10^{-680} \\
\mu_{2001,14} &= -2.89063\ldots \cdot 10^{-705} \\
\mu_{2001,15} &= -2.27283\ldots \cdot 10^{-726}
\end{aligned}
$$

More Notation

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,m} \qquad \nu_{N,n} = \sum_{m=1}^{n} \frac{\mu_{N,m}}{m}$$

$$\begin{aligned}
\mu_{2001,3}/3/\nu_{2001,2} &= -1 + 6.813\ldots \cdot 10^{-115} \\
\mu_{2001,4}/4/\nu_{2001,3} &= -1 - 2.533\ldots \cdot 10^{-98} \\
\mu_{2001,5}/5/\nu_{2001,4} &= -1 - 2.719\ldots \cdot 10^{-81} \\
\mu_{2001,6}/6/\nu_{2001,5} &= -1 + 5.188\ldots \cdot 10^{-67} \\
\mu_{2001,7}/7/\nu_{2001,6} &= -1 + 3.108\ldots \cdot 10^{-56} \\
\mu_{2001,8}/8/\nu_{2001,7} &= -1 - 7.566\ldots \cdot 10^{-48} \\
\mu_{2001,9}/9/\nu_{2001,8} &= -1 + 3.328\ldots \cdot 10^{-41} \\
\mu_{2001,10}/10/\nu_{2001,9} &= -1 - 2.131\ldots \cdot 10^{-36} \\
\mu_{2001,11}/11/\nu_{2001,10} &= -1 - 6.824\ldots \cdot 10^{-32} \\
\mu_{2001,12}/12/\nu_{2001,11} &= -1 - 7.437\ldots \cdot 10^{-28}
\end{aligned}$$

Striking Continued Fraction

$$\mu_{N,n} = \sum_{m|n} \mu\left(\frac{n}{m}\right) \delta_{N,m} \qquad \nu_{N,n} = \sum_{m=1}^{n} \frac{\mu_{N,m}}{m}$$

$$L = 2520 = \mathrm{LCM}(1,2,3,4,5,6,7,8,9,10) \qquad N = L + 1$$

$$2L \cdot \nu_{N,L} = 0.9998015873172093...$$

$$= \cfrac{1}{1+\cfrac{1}{5039+\cfrac{1}{2520+\cfrac{1}{1680+\cfrac{1}{1260+\cfrac{1}{1008+\cfrac{1}{840+\cfrac{1}{720+\cfrac{1}{630+\cfrac{1}{560+\cfrac{1}{504+\cfrac{1}{\ddots}}}}}}}}}}}}$$

$$5039 = 2L-1,\ 2520 = \tfrac{2L}{2},\ 1680 = \tfrac{2L}{3},\ 1260 = \tfrac{2L}{4},\ 1008 = \tfrac{2L}{5},$$
$$840 = \tfrac{2L}{6},\ 720 = \tfrac{2L}{7},\ 630 = \tfrac{2L}{8},\ 560 = \tfrac{2L}{9},\ 504 = \tfrac{2L}{10}$$

Function $\phi(L)$

$$\phi(L) = \frac{1}{2L} \cdot \cfrac{1}{1 + \cfrac{1}{2L - 1 + \cfrac{1}{\frac{2L}{2} + \cfrac{1}{\frac{2L}{3} + \cfrac{1}{\frac{2L}{4} + \cfrac{1}{\frac{2L}{5} + \cfrac{1}{\frac{2L}{6} + \ddots}}}}}}}$$

$$= \frac{1}{2}\psi\left(\frac{L}{2} + 1\right) - \frac{1}{2}\psi\left(\frac{L+1}{2}\right)$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

Other Values of N and L

$$\mu_{N,n} = \sum_{m|n} \mu\left(\tfrac{n}{m}\right) \delta_{N,m} \qquad \nu_{N,n} = \sum_{m=1}^{n} \frac{\mu_{N,m}}{m}$$

$$\phi(L) = \frac{1}{2}\psi\left(\frac{L}{2}+1\right) - \frac{1}{2}\psi\left(\frac{L+1}{2}\right) \qquad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

$$\frac{\nu_{2521,2520}}{\phi(2520)} = 1 - 1.063066513532\ldots \cdot 10^{-108}$$

$$\frac{\nu_{3001,3000}}{\phi(3000)} = 1 + 7.158776770618\ldots \cdot 10^{-128}$$

$$\frac{\nu_{6001,6000}}{\phi(6000)} = 1 + 5.411860996641659\ldots \cdot 10^{-259}$$

$$\frac{\nu_{7001,6000}}{\phi(6000)} = 1 + 5.258535208832606\ldots \cdot 10^{-209}$$

Lerch Function $\Phi(z, s, a)$

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$\Phi(1,s,1) = \zeta(s) \qquad \Phi(-1,s,1) = \eta(s) = (1 - 2 \cdot 2^{-s})\zeta(s)$$

$$\begin{aligned}
\Phi(-1,1,a+1) &= \phi(a) = \frac{1}{2}\psi\left(\frac{a}{2}+1\right) - \frac{1}{2}\psi\left(\frac{a+1}{2}\right) \\
&= \frac{1}{2a} \cdot \cfrac{1}{1 + \cfrac{1}{2a - 1 + \cfrac{1}{\frac{2a}{2} + \cfrac{1}{\frac{2a}{3} + \cfrac{1}{\frac{2a}{4} + \cfrac{1}{\frac{2a}{5} + \cfrac{1}{\frac{2a}{6} + \ddots}}}}}}}
\end{aligned}$$